Volume Conservation

Extending your 2D Hyper elastic simualtor

Kenny Erleben
Department of Computer Science
University of Copenhagen

Rate of Change of Volume Page 1

We know

$$dv = j dV$$

For incompressibility we always want

$$dv = dV \quad \Rightarrow \quad j = 1 \quad \text{and} \quad \frac{\partial j}{\partial t} = 0$$

The rate of change of volume is

$$\frac{\partial}{\partial t} dv = \frac{\partial j}{\partial t} dV = \frac{\frac{\partial j}{\partial t}}{j} dv$$

Rate of Change of Volume Page 2

One can compute j very fast assuming

- Deformation gradient is constant over triangle/tetrahedron e

Then in 2D we have

$$j^e = \frac{a^e}{A^e}$$

where a^e is the current spatial area of the triangle and A^e is the material area of the triangle.

In 3D we have

$$j^e = \frac{v^e}{V^e}$$

where v^e is the current spatial volume of the tetrahedron and V^e is the material volume of the tetrahedron.

Rate of Change of Volume Page 3

From chain rule we have([1])ß

$$\frac{\partial j}{\partial t} = \frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} : \frac{\partial \mathbf{F}}{\partial t}$$

We know $\dot{\mathbf{F}} = \mathbf{V}\mathbf{F}$ and $\frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} = \det(\mathbf{F})\mathbf{F}^{-T}$ so

$$\begin{aligned}\frac{\partial j}{\partial t} &= j\left(\mathbf{F}^{-T} : \mathbf{V}\mathbf{F}\right) = j\left(\mathbf{F}^{-T}\mathbf{F}^{T} : \mathbf{V}\right)\\ &= j\mathrm{tr}\left(\mathbf{V}\right) = j\mathrm{tr}\left(\nabla \mathbf{v}\right)\\ &= j\nabla \cdot v\end{aligned}$$

Observe that for incompressibility $j = 1$ and $\frac{\partial j}{\partial t} = 0$. So

$$j\nabla \cdot \mathbf{v} = 0$$

Rate of Change of Volume Page 3

Since

$$\mathrm{tr}\left(\nabla \mathbf{v}\right) = \mathrm{tr}\left(\mathbf{V}\right) = \mathrm{tr}\left(\frac{1}{2}\left(\mathbf{V} + \mathbf{V}^T\right)\right) = \mathrm{tr}\left(\mathbf{D}\right)$$

and $\mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$ we find

$$\frac{\partial j}{\partial t} = j\mathrm{tr}\left(\mathbf{V}\right) = j\mathrm{tr}\left(\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}\right)$$

And so

$$\frac{\partial j}{\partial t} = j\mathrm{tr}\left(\mathbf{C}^{-1}\right)\dot{\mathbf{E}} = j\mathbf{C}^{-1} : \dot{\mathbf{E}}$$

Finally giving

$$\frac{\partial j}{\partial t} = \frac{1}{2} j\mathbf{C}^{-1} : \dot{\mathbf{C}}$$

For incompressibility we have

Rate of Change of Mass Page 1

From

$$\rho_0 = j\rho$$

We have

$$\frac{d\rho_0}{dt} = \frac{\partial j}{\partial t}\rho + j\frac{d\rho}{dt}$$
$$0 = j\left(\nabla \cdot v\right)\rho + j\frac{d\rho}{dt}$$
$$0 = j\left(\nabla \cdot v\right)\rho + j\frac{d\rho}{dt}$$

Resulting in

$$\frac{d\rho}{dt} + \rho\left(\nabla \cdot v\right) = 0$$

Rate of Change of Mass Page 2

So we have

$$\frac{d\rho}{dt} + \rho \left(\nabla \cdot v \right) = 0$$

Now $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v}$ so

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla \rho \cdot \mathbf{v} + \rho \left(\nabla \cdot v \right)}_{\nabla \cdot (\rho \mathbf{v})} = 0$$

That is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

For incompressibility $\frac{\partial \rho}{\partial t} = 0$ and homogenous material $\nabla \rho = \mathbf{0}$. This boils down to

In Summary

Applying the incompressibility assumption means

$$\frac{\partial j}{\partial t} = 0$$

The rate of change of volume and mass results in the relations

$$j \nabla \cdot \mathbf{v} = 0$$

$$\frac{1}{2} j \mathbf{C}^{-1} : \dot{\mathbf{C}} = 0$$

Note, for incompressibility $j = 1$ but we keep j in our equations.

The Idea

What we will do in the following is basically to split the distortion term of deformation from the volume change term

Distortional Component of Deformation Gradient

By definition the distortional (Danish: "forvridende") component $\hat{\mathbf{F}}$ does not imply any change in volume. Hence

$$\det \hat{\mathbf{F}} = 1$$

Defining $\hat{\mathbf{F}} = j^{-\frac{1}{3}}\mathbf{F}$ where $j = \det \mathbf{F}$

$$\det \hat{\mathbf{F}} = \det\left(j^{-\frac{1}{3}}\mathbf{F}\right) = \left(j^{-\frac{1}{3}}\right)^3 \det \mathbf{F} = j^{-1} j = 1$$

The deformation gradient is

$$\mathbf{F} = j^{\frac{1}{3}}\hat{\mathbf{F}}$$

The Distortional Strain Tensors

We have

$$\hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}}$$

Observe

$$\hat{\mathbf{C}} = \left(j^{-\frac{1}{3}}\right)^2 \mathbf{F}^T \mathbf{F} = \left(j^2\right)^{-\frac{1}{3}} \mathbf{C} = (\det \mathbf{C})^{-\frac{1}{3}} \mathbf{C}$$

And

$$\hat{\mathbf{E}} = \frac{1}{2}\left(\hat{\mathbf{C}} - \mathbf{I}\right)$$

The Deviatoric Stress Page 1

The deviatoric (Danish "snyde") stress component, σ' is defined such that

$$\sigma = \sigma' + p\mathbf{I}$$

where p is defined as $p = \frac{1}{3}\mathrm{tr}\left(\sigma\right) = \frac{1}{3}\sigma : \mathbf{I}$. Further

$$\mathrm{tr}\left(\sigma'\right) = \mathrm{tr}\left(\sigma - p\mathbf{I}\right) = \mathrm{tr}\left(\sigma\right) - p\mathrm{tr}\left(\mathbf{I}\right) = \mathrm{tr}\left(\sigma\right) - 3\,p = 0$$

Using $\mathbf{P} = j\sigma\mathbf{F}^{-T}$ and $\mathbf{S} = j\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}$ the deviatoric counter parts $\mathbf{P}'$ and $\mathbf{S}'$ are defined as

$$\mathbf{P}' = j\sigma'\mathbf{F}^{-T}$$
$$\mathbf{S}' = j\mathbf{F}^{-1}\sigma'\mathbf{F}^{-T}$$

Observe we have $\mathbf{P}' = \mathbf{F}\mathbf{S}'$

The Deviatoric Stress Page 2

From

$$\mathbf{P} = j\sigma\mathbf{F}^{-T} = j\left(\sigma' + p\mathbf{I}\right)\mathbf{F}^{-T}$$
$$\mathbf{S} = j\mathbf{F}^{-1}\sigma\mathbf{F}^{-T} = j\mathbf{F}^{-1}\left(\sigma' + p\mathbf{I}\right)\mathbf{F}^{-T}$$

We have

$$\mathbf{P} = \mathbf{P}' + pj\mathbf{F}^{-T}$$
$$\mathbf{S} = \mathbf{S}' + pj\mathbf{C}^{-1}$$

The Hydrostatic Pressure Page 1

From the deviatoric stress definitions we have

$$\mathbf{P}' : \mathbf{F} = \text{tr}\left(j\sigma'\mathbf{F}^{-T}\mathbf{F}^{T}\right) = j\text{tr}\left(\sigma'\right) = 0$$
$$\mathbf{S}' : \mathbf{C} = \text{tr}\left(j\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}\mathbf{C}\right) = j\text{tr}\left(\sigma'\right) = 0$$

So

$$\mathbf{P}' : \mathbf{F} = 0$$
$$\mathbf{S}' : \mathbf{C} = 0$$

This means

$$\mathbf{P} : \mathbf{F} = \mathbf{P}' : \mathbf{F} + pj\,\mathbf{F}^{-T} : \mathbf{F} = p\,j\,\text{tr}\left(\mathbf{F}^{-T}\mathbf{F}\right) = 3\,j\,p$$
$$\mathbf{S} : \mathbf{C} = \mathbf{S}' : \mathbf{C} + pj\,\mathbf{C}^{-1} : \mathbf{C} = p\,j\,\text{tr}\left(\mathbf{C}^{-1}\mathbf{C}\right) = 3\,j\,p$$

The Hydrostatic Pressure Page 2

From

$$\mathbf{P} : \mathbf{F} = 3\, j\, p$$
$$\mathbf{S} : \mathbf{C} = 3\, j\, p$$

We obtain

$$p = \frac{1}{3} j^{-1} \mathbf{P} : \mathbf{F}$$
$$p = \frac{1}{3} j^{-1} \mathbf{S} : \mathbf{C}$$

Incompressible Elasticity Page 1

From time derivative of strain energy

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial \mathbf{C}} : \frac{\partial \mathbf{C}}{\partial t} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}}$$

So from this we find by rearranging terms

$$\left(\frac{1}{2}\mathbf{S} - \frac{\partial \psi}{\partial \mathbf{C}} \right) : \dot{\mathbf{C}} = 0$$

For incompressibility $\frac{\partial j}{\partial t} = 0$ this results in the constraint

$$\frac{1}{2} j \, \mathbf{C}^{-1} : \dot{\mathbf{C}} = 0$$

Comparing the two equations this implies

$$\left(\frac{1}{2}\mathbf{S} - \frac{\partial \psi}{\partial \mathbf{C}} \right) = \gamma \frac{1}{2} j \, \mathbf{C}^{-1}$$

Incompressible Elasticity Page 2

From

$$\left(\frac{1}{2}\mathbf{S} - \frac{\partial \psi}{\partial \mathbf{C}}\right) = \gamma \frac{1}{2} j\, \mathbf{C}^{-1}$$

The general incompressible hyperelastic constitutive equation is

$$\mathbf{S} = 2\frac{\partial \psi(\mathbf{C})}{\partial \mathbf{C}} + \gamma j\, \mathbf{C}^{-1}$$

Comparing with

$$\mathbf{S} = \mathbf{S}' + pj\mathbf{C}^{-1}$$

Suggest a relationship between p and γ

Incompressible Elasticity Page 3

From $\mathbf{S} = \mathbf{S}' + pj\mathbf{C}^{-1}$ we have

$$p = \frac{1}{3} j^{-1} \mathbf{S} : \mathbf{C}$$

Inserting our formula for $\mathbf{S}$

$$p = \frac{1}{3} j^{-1} \left(2 \frac{\partial \psi(\mathbf{C})}{\partial \mathbf{C}} + \gamma j \, \mathbf{C}^{-1} \right) : \mathbf{C}$$

we have

$$p = \gamma + \frac{2}{3} j^{-1} \frac{\partial \psi(\mathbf{C})}{\partial \mathbf{C}} : \mathbf{C}$$

So p and γ coincides if

$$\frac{\partial \psi(\mathbf{C})}{\partial \mathbf{C}} : \mathbf{C} = 0$$

Incompressible Elasticity Page 4

The property $\frac{\partial \psi(\mathbf{C})}{\partial \mathbf{C}} : \mathbf{C} = 0$ is fufilled if $\psi(\alpha \mathbf{C}) = \psi(\mathbf{C})$ for arbitary constant α. Using $\hat{\psi}(\mathbf{C}) = \psi(\hat{\mathbf{C}})$

$$\begin{aligned}\hat{\psi}(\alpha \mathbf{C}) &= \psi(\det(\alpha \mathbf{C})^{-\frac{1}{3}}(\alpha \mathbf{C})) \\ &= \psi(\det(\mathbf{C})^{-\frac{1}{3}}\mathbf{C}) \\ &= \psi(\hat{\mathbf{C}}) \\ &= \hat{\psi}(\mathbf{C})\end{aligned}$$

Thus, we have

$$\mathbf{S} = 2\frac{\partial \hat{\psi}(\mathbf{C})}{\partial \mathbf{C}} + pj\,\mathbf{C}^{-1}$$

and

$$\ldots \frac{\partial \hat{\psi}(\mathbf{C})}{\ldots}$$

In Summary

Adding the constraint $j = 1$ or equivalently $\frac{\partial j}{\partial t} = 0$ means we have to use the constitutive equations

$$\mathbf{S} = 2\frac{\partial \hat{\psi}(\mathbf{C})}{\partial \mathbf{C}} + pj\,\mathbf{C}^{-1}$$

and

$$\mathbf{S}' = 2\frac{\partial \hat{\psi}(\mathbf{C})}{\partial \mathbf{C}}$$

where $\hat{\psi}(\mathbf{C}) = \psi(\hat{\mathbf{C}})$.

This was theory now let us develop a numerical method

Putting Things together

The Cauchy equation, deviatoric stress tensor and volume constraint

$$\rho \ddot{\mathbf{x}} = \mathbf{b} + \nabla \cdot \sigma$$
$$\sigma = \sigma' + p\mathbf{I}$$
$$\nabla \cdot \mathbf{v} = 0$$

Doing FVM discretization leads to $\mathbf{v}'_i = \mathbf{v}^n + \frac{\Delta t}{m_i} \mathbf{f}_i^{\text{total}}$ (^{2}) and

$$\mathbf{v}_i^{n+1} = \mathbf{v}' + \frac{\Delta t}{m_i} \int_{a_i} \nabla \cdot p\mathbf{I} da$$
$$\mathbf{x}^{n+1} = \mathbf{x}^n + \Delta t \mathbf{v}^{n+1}$$

and requiring volume constraints to be fulfilled at end of time step

$$\int \nabla \cdot \mathbf{v}^{n+1} da = 0$$

What is Missing?

We need discrete counter parts of the terms

$$\int_{a_i} \nabla \cdot \mathbf{v}^{n+1} da \approx \mathbf{B}_{i,1..N} \begin{bmatrix} \mathbf{v}_1^{n+1} \\ \vdots \\ \mathbf{v}_N^{n+1} \end{bmatrix}$$

$$\frac{\Delta t}{m_i} \int_{a_i} \nabla \cdot p\mathbf{I} da \approx \mathbf{A}_{i,1..N} \begin{bmatrix} p_1^{n+1} \\ \vdots \\ p_N^{n+1} \end{bmatrix}$$

We will develop these terms $\mathbf{A}_{i,1..N}$ and $\mathbf{B}_{i,1..N}$ later.

Observe the dimentions of the sub-blocks $\mathbf{A}_{ij} \in \mathbb{R}^{2\times 1}$ and $\mathbf{B}_{ij} \in \mathbb{R}^{1\times 2}$.

Continuing the Story

Assume $\mathbf{A}_{i,1..N}$ and $\mathbf{B}_{i,1..N}$ are known then we stack all control volumes

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1..N} \\ \vdots \\ \mathbf{A}_{N,1..N} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1..N} \\ \vdots \\ \mathbf{B}_{N,1..N} \end{bmatrix}$$

So we have

$$\underbrace{\begin{bmatrix} \mathbf{v}_1^{n+1} \\ \vdots \\ \mathbf{v}_N^{n+1} \end{bmatrix}}_{\mathbf{w}^{n+1}} = \underbrace{\begin{bmatrix} \mathbf{v}'_1 \\ \vdots \\ \mathbf{v}'_N \end{bmatrix}}_{\mathbf{w}'} + \mathbf{A} \underbrace{\begin{bmatrix} p_1^{n+1} \\ \vdots \\ p_N^{n+1} \end{bmatrix}}_{\mathbf{q}}$$

That is

$$\mathbf{w}^{n+1} = \mathbf{w}' + \mathbf{A}\mathbf{q} \quad \text{and} \quad \mathbf{B}\mathbf{w} = \mathbf{0}$$

Solving the Volume Constraint

We want volume to be conserved at end of time-step so

$$\mathbf{B}\mathbf{w}^{n+1} = \mathbf{B}\mathbf{w}' + \underbrace{\mathbf{B}\mathbf{A}}_{\mathbf{H}}\mathbf{q} = \mathbf{0}$$

Thus, we solve

$$\mathbf{H}\mathbf{q} = -\mathbf{B}\mathbf{w}'$$

Once $\mathbf{q}$ is known we can compute final velocities

$$\mathbf{w}^{n+1} = \mathbf{w}' + \mathbf{A}\mathbf{q}$$

And finally the position update

$$\begin{bmatrix} \mathbf{x}_1^{n+1} \\ \vdots \\ \ldots \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^{n} \\ \vdots \\ \ldots \end{bmatrix} + \Delta t \mathbf{w}^{n+1}$$

Discretization Preliminaries Page 1

We will use a FEM approximation to interpolate values. Assume $\mathbf{x}$ lies in the triangle e made of vertices i, j, k.

$$\mathbf{v}(\mathbf{x}) = \sum_{a \in \{i,j,k\}} \omega_a(\mathbf{x}) \mathbf{v}_a$$

$$p(\mathbf{x}) = \sum_{a \in \{i,j,k\}} \omega_a(\mathbf{x}) p_a$$

where

$$\omega_i(\mathbf{x}) = \frac{\frac{1}{2}(\mathbf{x} - \mathbf{x}_k) \times (\mathbf{x}_j - \mathbf{x}_k)}{\frac{1}{2}(\mathbf{x}_i - \mathbf{x}_k) \times (\mathbf{x}_j - \mathbf{x}_k)} = \frac{(\mathbf{x} - \mathbf{x}_k) \times (\mathbf{x}_j - \mathbf{x}_k)}{2a_e}$$

where we defined $a_e = \frac{1}{2}(\mathbf{x}_i - \mathbf{x}_k) \times (\mathbf{x}_j - \mathbf{x}_k)$. Now

$$\nabla \omega_i(\mathbf{x}) = -\frac{1}{2a_e} \mathbf{n}_i l_i$$

Discretization Preliminaries Page 2

So

$$\nabla \cdot \mathbf{v}(\mathbf{x}) = \sum_{a \in \{i,j,k\}} \nabla \omega_a(\mathbf{x}) \cdot \mathbf{v}_a$$

$$\nabla p(\mathbf{x}) = \sum_{a \in \{i,j,k\}} \nabla \omega_a(\mathbf{x}) p_a$$

and we have

$$\nabla \cdot \mathbf{v}(\mathbf{x}) = -\frac{1}{2a_e} \sum_{a \in \{i,j,k\}} \mathbf{n}_a l_a \cdot \mathbf{v}_a$$

$$\nabla p(\mathbf{x}) = -\frac{1}{2a_e} \sum_{a \in \{i,j,k\}} \mathbf{n}_a l_a p_a$$

Discretization Divergence Term

We have the median dual centered control volume a_i

- Let $\mathcal{T}_i$ be the index set of all triangles of the control volume
- and let $\mathcal{V}_e$ be the node index set of the e^{th} triangle

$$\int_{a_i} \nabla \cdot \mathbf{v}^{n+1} da = \sum_{e \in T_i} \frac{1}{3} \int_{a_e} \nabla \cdot \mathbf{v}^{n+1} da$$
$$= - \sum_{e \in T_i} \sum_{a \in \mathcal{V}_e} \frac{1}{6} \mathbf{n}_a l_a \cdot \mathbf{v}_a^{n+1}$$

Thus,

$$\mathbf{B}_{i,1..N} \mathbf{w}^{n+1} = -\frac{1}{6} \sum_{e \in T_i} \sum_{a \in \mathcal{V}_e} \mathbf{n}_a l_a \cdot \mathbf{v}_a^{n+1}$$

Discretization Pressure Term

For the i^{th} control volume

$$\frac{\Delta t}{m_i}\int_{a_i} \nabla \cdot p\mathbf{I} da = \frac{\Delta t}{m_i}\sum_{e \in T_i}\frac{1}{3}\int_{a_e} \nabla p da$$
$$= -\frac{\Delta t}{m_i}\sum_{e \in T_i}\sum_{a \in \mathcal{V}_e}\frac{1}{6}\mathbf{n}_a l_a p_a$$

Thus,

$$\mathbf{A}_{i,1..N}\mathbf{q}^{n+1} = -\frac{\Delta t}{m_i}\frac{1}{6}\sum_{e \in T_i}\sum_{a \in \mathcal{V}_e}\mathbf{n}_a l_a p_a$$

In Summary

For the i^{th} control volume

$$\mathbf{B}_{ij} = -\frac{1}{6}\begin{cases} \sum_e l_j \mathbf{n}_j^T & \text{If } j \in \mathcal{V}_e \text{ and } e \in \mathcal{T}_i \\ \mathbf{0}^T & \text{otherwise} \end{cases}$$

and so we have

$$\mathbf{A}_{ij} = -\frac{1}{6}\frac{\Delta t}{m_i}\begin{cases} \sum_e l_j \mathbf{n}_j & \text{If } j \in \mathcal{V}_e \text{ and } e \in \mathcal{T}_i \\ \mathbf{0}^T & \text{otherwise} \end{cases}$$

Matrix Properties of **H**

Now **H** is given by

$$\mathbf{H}_{ij} = \sum_{k=1}^{N} \mathbf{B}_{ik}\mathbf{A}_{k,j}$$

- **H** is a square matrix
- **H** has symmetic fill pattern but **H** is NOT a symmetric matrix
- **H** is close to singular

Empty Space

Empty Space

A Variational Approach Page 1

Applying variational principle to derive FEM equations one solves

$$D\Pi(\phi)[\mathbf{w}] = 0$$

here $\Pi(\phi)$ is total potential energy functional and $\mathbf{w}$ is virtual velocity

$$\Pi(\phi) \equiv \int_V \psi(\mathbf{C}) - \int_V \mathbf{f}\cdot\phi dV - \int_\Gamma \mathbf{t}\cdot\phi d\Gamma$$

Using $\psi(\mathbf{C}) = \hat{\psi}(\mathbf{C}) + U(j)$

$$\Pi(\phi) \equiv \underbrace{\int_V \hat{\psi}(\mathbf{C})dV - \int_V \mathbf{f}\cdot\phi dV - \int_\Gamma \mathbf{t}\cdot\phi d\Gamma}_{\equiv\hat{\Pi}(\phi)} + \underbrace{\int_V U(j)dV}_{\equiv\Pi_{\text{vol}}(\phi)}$$

A Variational Approach Page 2

We now have that our FEM equations are given by

$$D\Pi(\phi)[\mathbf{w}] = D\hat{\Pi}(\phi)[\mathbf{w}] + D\Pi_{\text{vol}}(\phi)[\mathbf{w}] = 0$$

We already know how to deal with the term $D\hat{\Pi}(\phi)[\mathbf{w}]$ so we only care about $D\Pi_{\text{vol}}(\phi)[\mathbf{w}]$

$$D\Pi_{\text{vol}}(\phi)[\mathbf{w}] = \int_V DU(j)[\mathbf{w}]dV$$

Directional Derivatives Page 1

By definition $j(\mathbf{x}) = \det(\mathbf{F}(\mathbf{x}))$ so by chain rule we have

$$Dj[\mathbf{u}] = \frac{\partial j(\mathbf{F})}{\partial \mathbf{F}} : D\mathbf{F}[\mathbf{u}]$$

We already know $\frac{\partial j(\mathbf{F})}{\partial \mathbf{F}} = j\,\mathbf{F}^{-T}$ and that $D\mathbf{F}[\mathbf{u}] = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \nabla \mathbf{u}\,\mathbf{F}$ and so

$$Dj[\mathbf{u}] = j\,\mathbf{F}^{-T} : \nabla \mathbf{u}\,\mathbf{F}$$

which we can clean up

$$\begin{aligned} Dj[\mathbf{u}] &= j\,\mathbf{F}^{-T}\mathbf{F}^{T} : \nabla \mathbf{u} \\ &= j\,\mathbf{I} : \nabla \mathbf{u} \\ &= j\,\mathrm{tr}\left(\nabla \mathbf{u}\right) = j \nabla \cdot \mathbf{u} \end{aligned}$$

Leaving us with

Directional Derivatives Page 2

Imagine we have some function $U(j)$ and we want to compute

$$DU(j)[\mathbf{u}] = \frac{\partial U}{\partial j}\frac{\partial j}{\partial \mathbf{F}} : D\mathbf{F}[\mathbf{u}]$$

Which boils down to

$$DU(j)[\mathbf{u}] = \frac{\partial U}{\partial j} j \, \nabla \cdot \mathbf{u}$$

Given the specific function

$$U(j) = \frac{1}{2}\kappa(j-1)^2$$

we find $\frac{\partial U}{\partial j} = \kappa\,(j-1)$ and so

$$DU(j)[\mathbf{u}] = \kappa\,(j-1)\,j\,\nabla \cdot \mathbf{u}$$

The Volumetric Variational Term

Thus, putting things together we have now found that

$$D\Pi_{\text{vol}}(\phi)[\mathbf{w}] = \int_V \frac{\partial U}{\partial j} j \, \nabla \cdot \mathbf{w} dV$$

Introducing a set of elements we have

$$D\Pi_{\text{vol}}(\phi)[\mathbf{w}] = \sum_e D\Pi^e_{\text{vol}}(\phi)[\mathbf{w}]$$

Assuming we are in 2D and that **F** and j are uniform over an element e we have

$$D\Pi^e_{\text{vol}}(\phi)[\mathbf{w}] = A^e \, p^e \, j^e \, \nabla \cdot \mathbf{w} = a^e \, p^e \, \nabla \cdot \mathbf{w}$$

we used $p^e = \frac{\partial U(j^e)}{\partial j}$ and $j^e = \frac{a^e}{A^e}$ with a^e being spatial area and A^e material area.

The Discrete Volumetric Variational Term Page 1

We have

$$D\Pi^e_{\text{vol}}(\phi)[\mathbf{w}] = a^e\, p^e\, \nabla \cdot \mathbf{w}$$

Assume the standard FEM approximation then

$$\mathbf{w} = \sum_{a \in \mathcal{V}_e} = \omega_a(\mathbf{x})\mathbf{w}^e_a$$

Thus

$$\nabla \cdot \mathbf{w} = \sum_{a \in \mathcal{V}_e} = \nabla\omega_a(\mathbf{x}) \cdot \mathbf{w}^e_a$$

And

$$D\Pi^e_{\text{vol}}(\phi)[\mathbf{w}] = \sum_{a \in \mathcal{V}_e} a^e\, p^e\, \nabla\omega_a(\mathbf{x}) \cdot \mathbf{w}^e_a$$

The Discrete Volumetric Variational Term Page 2

This means

$$
\begin{aligned}
D\Pi_{\text{vol}}(\phi)[\mathbf{w}] &= \sum_e D\Pi^e_{\text{vol}}(\phi)[\mathbf{w}] \\
&= \sum_e \sum_{a \in \mathcal{V}_e} a^e\, p^e\, \nabla \omega_a(\mathbf{x}) \cdot \mathbf{w}^e_a \\
&= \underbrace{\sum_a \left(\sum_{e, a \in \mathcal{V}_e} \underbrace{a^e(\mathbf{x})\, p^e\, \nabla \omega_a(\mathbf{x})}_{\mathbf{h}^e_a} \right)}_{\mathbf{h}(\phi, \mathbf{p})} \cdot \mathbf{w}_a
\end{aligned}
$$

This essentially means the equations of motion gets and extra nonlinear volume term

$$
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{k}(\mathbf{x} - \mathbf{X}) + \mathbf{h}(\mathbf{x}, \mathbf{p}) = \mathbf{f}
$$

Revisiting Strain and Consitutive Equation Page 1

Defining the strain energy as $\psi(\mathbf{C}) = \hat{\psi}(\mathbf{C}) + U(j)$ and using $\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}}$ we find

$$\mathbf{S} = 2\frac{\partial \hat{\psi}}{\partial \mathbf{C}} + 2\frac{\partial U(j)}{\partial \mathbf{C}}$$

using deviatoric stress tensor and chain rule simplifies to

$$\mathbf{S} = \mathbf{S}' + 2\frac{\partial U}{\partial j}\frac{\partial j}{\partial \mathbf{C}}$$

We know $j^2 = III_C$ and $\frac{\partial III_C}{\partial \mathbf{C}} = j^2\, C^{-1}$. From chain rule

$$\frac{\partial j^2}{\partial \mathbf{C}} = 2\, j\, \frac{\partial j}{\partial \mathbf{C}}$$

so $\frac{\partial j}{\partial \mathbf{C}} = \frac{1}{2}\, j\, \mathbf{C}^{-1}$ and

$$\mathbf{S} = \mathbf{S}' + \partial U \ldots \mathbf{C}^{-1}$$

Revisiting Strain and Consitutive Equation Page 2

We know from previously that

$$\mathbf{S} = \mathbf{S}' + p\, j\, \mathbf{C}^{-1}$$

Comparing with

$$\mathbf{S} = \mathbf{S}' + \frac{\partial U}{\partial j}\, j\, \mathbf{C}^{-1}$$

Gives

$$p = \frac{\partial U}{\partial j}$$

For the case of $U(j) = \frac{1}{2}\,\kappa\,(j-1)^2$ one finds

$$p = \kappa\,(j-1)$$

The Variational Numerical Method

A single simulation step:

- Step 1 Compute $\hat{\mathbf{F}}^e$ for all e (Distortional)
- Step 2 Compute $\hat{\mathbf{E}}^e$ for all e (Distortional)
- Step 3 Compute $\mathbf{S}'^e$ for all e (Deviatoric)
- Step 4 Compute $\mathbf{P}'^e$ for all e (Deviatoric)
- Step 5a Compute elastic forces from deviatoric stress
- Step 5b Compute hydrostatic pressure p^e for all e
- Step 5c Compute hydrostatic pressure forces $\mathbf{h}_i^e = p^e a^e \nabla w_i$ for all i
- Step 6 Compute total forces $\mathbf{f}_i^{\text{total}}$ for all i
- Step 7 Compute velocity update $\mathbf{v}_i^{t+\Delta t}$ for all i
- Step 8 Compute position update $\mathbf{x}_i^{t+\Delta t}$ for all i
- Step 9 Increment time $t \leftarrow t + \Delta t$

Empty Space

The Average Nodal Pressure Formulation Page 1

The main idea is to assume that j is constant over the control volume of the entire node

$$j_i = \sum_{e \in \mathcal{N}_i} \frac{a^e}{A^e}$$

Using this FVM control volume approach we have

$$\Pi_{\text{vol}}(\mathbf{x}) = \sum_i U(j_i) A_i$$

The variational principle

$$\Pi_{\text{vol}}(\mathbf{x})[\mathbf{w}] = \sum_i p_i A_i D j_i[\mathbf{w}]$$

where the average nodal pressure is given as

$$a_i - A_i$$

The Average Nodal Pressure Formulation Page 2

Using $j_i = a_i / A_i$ the directional derivative is rewritten as

$$Dj_i[\mathbf{w}] = \frac{1}{A_i} Da_i[\mathbf{w}] = \frac{1}{A_i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} Da^e[\mathbf{w}]$$

Further we find

$$\frac{1}{A_i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} Da^e[\mathbf{w}] = \frac{1}{A_i} \sum_{e \in \mathcal{N}_i} A^e \frac{1}{3} Dj^e[\mathbf{w}]$$

and

$$\frac{1}{A_i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} A^e Dj^e[\mathbf{w}] = \frac{1}{A_i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} A^e j^e \nabla \cdot \mathbf{w} = \frac{1}{A_i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} a^e \nabla \cdot \mathbf{w}$$

The Average Nodal Pressure Formulation Page 3

Putting things together we then have

$$D\Pi_{\text{vol}}(\mathbf{x})[\mathbf{w}] = \sum_{i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} p_i a^e \nabla \cdot \mathbf{w}$$

Applying the FEM approximation we have

$$D\Pi_{\text{vol}}(\mathbf{x})[\mathbf{w}] = \sum_{i} \sum_{e \in \mathcal{N}_i} \frac{1}{3} p_i a^e \sum_{a \in \mathcal{V}_e} \nabla \omega_a^e \cdot \mathbf{w}_a^e$$

Then we define $p^e = \sum_{e \in \mathcal{N}_i} \frac{1}{3} p_i^e$ and

$$D\Pi_{\text{vol}}(\mathbf{x})[\mathbf{w}] = \sum_{i} p^e \sum_{e \in \mathcal{N}_i} a^e \sum_{a \in \mathcal{V}_e} \nabla \omega_a^e \cdot \mathbf{w}_a^e$$