Revisiting Boundary Conditions

A short Note on Matrix Details

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The Boundary Conditions in Matrix Form

Assume a discrete model

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where $\mathbf{u}, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Now we wish to apply Dirichlet boundary conditions

$$\mathcal{D} \equiv \{ i | \mathbf{u}_i = d_i \}$$

then

$$\mathcal{F} \equiv \{ i | i \notin \mathcal{D} \}$$

Make partitioning of the original system

$$\begin{bmatrix} \mathbf{A}_{\mathcal{F}\mathcal{F}} & \mathbf{A}_{\mathcal{F}\mathcal{D}} \\ \mathbf{A}_{\mathcal{D}\mathcal{F}} & \mathbf{A}_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\mathcal{F}} \\ \mathbf{u}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{\mathcal{F}} \\ \mathbf{b}_{\mathcal{D}} \end{bmatrix}$$

Inserting the Boundary Condition

Insert Dirichlet boundary conditions

$$\mathbf{I}\mathbf{u}_{\mathcal{D}} = \mathbf{d}$$

Into partitioned system

$$\begin{bmatrix} \mathbf{A}_{\mathcal{F}\mathcal{F}} & \mathbf{A}_{\mathcal{F}\mathcal{D}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\mathcal{F}} \\ \mathbf{u}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{\mathcal{F}} \\ \mathbf{d} \end{bmatrix}$$

Resulting reduced system

$$\mathbf{A}_{\mathcal{F}\mathcal{F}}\mathbf{u}_{\mathcal{F}} = \mathbf{b}_{\mathcal{F}} - \mathbf{A}_{\mathcal{F}\mathcal{D}}\mathbf{d}$$

Notice that boundary conditions end up as an extra term on the right-hand side (extra source term)

Natural boundary Conditions

Consider von Neumann type of boundary conditions

$$\mathcal{N} \equiv \{i | j \notin i \wedge c_j \mathbf{u}_j - d_i \mathbf{u}_i = \mathbf{e}_i\}$$

then

$$\mathcal{F} \equiv \{i | i \notin \mathcal{N}\}$$

In matrix form

$$\mathbf{C}\mathbf{u}_{\mathcal{F}} + \mathbf{D}\mathbf{u}_{\mathcal{N}} = \mathbf{e}$$

Into partitioned system

$$\begin{bmatrix} \mathbf{A}_{\mathcal{F}\mathcal{F}} & \mathbf{A}_{\mathcal{F}\mathcal{N}} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\mathcal{F}} \\ \mathbf{u}_{\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{\mathcal{F}} \\ \mathbf{e} \end{bmatrix}$$

Natural boundary Conditions

The resulting reduced system (does anybody know what this is?)

$$\left(\mathbf{A}_{\mathcal{F}\mathcal{F}} - \mathbf{A}_{\mathcal{F}\mathcal{N}}\mathbf{D}^{-1}\mathbf{C}\right)\mathbf{u}_{\mathcal{F}} = \mathbf{b}_{\mathcal{F}} - \mathbf{A}_{\mathcal{F}\mathcal{N}}\mathbf{D}^{-1}\mathbf{e}$$

A little bit different than Dirichlet but has the same story