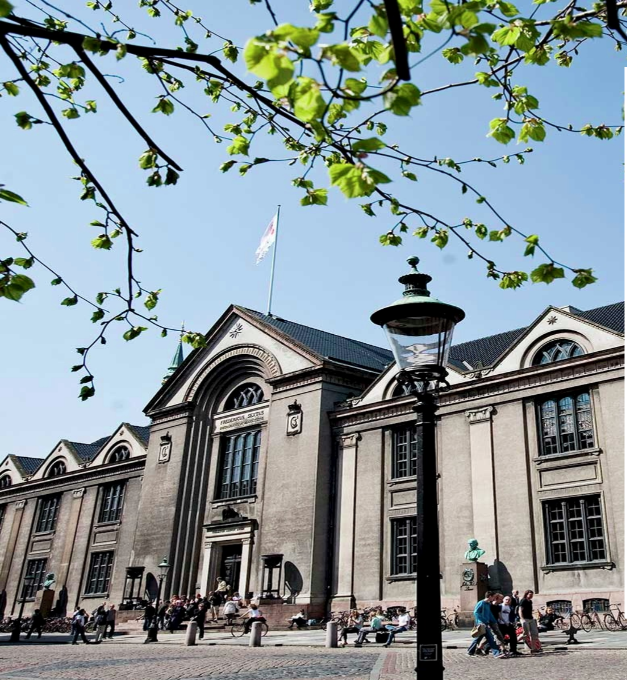

Strain and Stress Tensors

Introduction to Concept and Definitions

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Modeling of Elastic Solids

The partial differential equation, Cauchy Equation, gives the equation of motion of an elastic solid

$$\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \nabla \cdot \sigma$$

The boundary conditions of the Cauchy stress tensor σ are given by

$$\sigma \boldsymbol{n} = \boldsymbol{t}$$

where $\boldsymbol{t}$ is a known applied surface traction. Finally, a constitutive law is needed that provides a relationship between stress and strain

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}}$$

where $\mathbf{S}$ is 2. Piola–Kirchhoff stress tensor and $\mathbf{E}$ is the Green strain tensor. Now we will derive all these equations that we need.

Kinematics – Learning The Definitions of The Strain Tensors

The Material and Spatial Coordinates

We have two sets of coordinates

- Let $\boldsymbol{X}$ be undeformed coordinates
- and $\boldsymbol{x}$ the deformed coordinates

The deformation is given by the mapping from undeformed coordinates into deformed coordinates,

$$\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}, t)$$

Observe

- The same global reference coordinate system is used for both sets of coordinates.
- Undeformed coordinates are sometimes also called material coordinates and deformed coordinates are the spatial coordinates.

The Deformation Gradient

By definition of differentials we have

$$d\boldsymbol{x} = \underbrace{\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{X}}}_{\mathbf{F}} d\boldsymbol{X}$$

The partial derivative $\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{X}} = \mathbf{F}$ is referred to as the Deformation gradient.

In particular, we have

$$\mathbf{F}_{ij} = \frac{\partial \boldsymbol{\Phi}_i}{\partial \boldsymbol{X}_j} = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{X}_j}$$

Investigating Local Deformation Measures

Think of $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ as **arbitary** chosen small “needles”. By definition we have

$$d\boldsymbol{x}_1 = \mathbf{F} d\boldsymbol{X}_1$$
$$d\boldsymbol{x}_2 = \mathbf{F} d\boldsymbol{X}_2$$

To describe any local deformation we investigate the dot products

$$d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 \qquad \text{and} \qquad d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2$$

as these hold information about any local angle or length deformations.

The Right Cauchy–Green Deformation Tensor

First we will try to express $d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2$ in terms of material coordinates,

$$\begin{aligned} d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 &= d\boldsymbol{x}_1^T d\boldsymbol{x}_2 \\ &= \left(\mathbf{F} d\boldsymbol{X}_1\right)^T \left(\mathbf{F} d\boldsymbol{X}_2\right) \\ &= d\boldsymbol{X}_1^T \underbrace{\left(\mathbf{F}^T \mathbf{F}\right)}_{\mathbf{C}} d\boldsymbol{X}_2 \end{aligned}$$

where

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

is called the right Cauchy–Green Deformation tensor.

Tensor Invariants (1/2)

The eigenvalues of **C** are independent of rotation thus the coefficients of the characteristic polynomial are invariants

$$\det(\mathbf{C} - \mathbf{I}\lambda) = -\lambda^3 + \mathsf{I}_C\lambda^2 - \mathsf{II}_C^*\lambda + \mathsf{III}_C = 0$$

where

$$\begin{aligned} \mathsf{I}_C &= \operatorname{tr}(\mathbf{C}) = \mathbf{C} : \mathbf{I} \\ \mathsf{II}_C^* &= \frac{1}{2}\left(\operatorname{tr}(\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\right) \\ \mathsf{III}_C &= \det(\mathbf{C}) \end{aligned}$$

Tensor Invariants (2/2)

We prefer an alternative definition of the second invariant

$$\mathsf{II}_C = \mathrm{tr}\left(\mathbf{C}^2\right) = \mathbf{C} : \mathbf{C}$$

as it simplifies later equations

The Eigenvalue Decomposition

Since **C** is a symmetric positive definite we know that an eigenvalue decomposition exists

$$\mathbf{CN} - \mathbf{N} = 0$$

where

$$\mathbf{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 & \boldsymbol{N}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

and $\mathbf{N}^T\mathbf{N} = \mathbf{N}\mathbf{N}^T = \mathbf{I}$ and $0 < \lambda_3^2 \leq \lambda_2^2 \leq \lambda_1^2 \in \mathbb{R}$. This can be rewritten as

$$\mathbf{C} = \mathbf{N}\mathbf{N}^T = \sum \lambda_i^2 \left(\boldsymbol{N}_i \boldsymbol{N}_i^T \right)$$

Invariants in terms of Eigenvalues

Observe that knowing the eigenvalues λ_i^2 of **C** we can compute the tensor invariants as

$$
\begin{aligned}
\mathrm{I}_C &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
\mathrm{II}_C &= \lambda_1^4 + \lambda_2^4 + \lambda_3^4 \\
\mathrm{III}_C &= \lambda_1^2 \lambda_2^2 \lambda_3^2
\end{aligned}
$$

These formulas will become handy when we later wish to change coordinates.

A Rotation Tensor

The square root of C is

$$\mathbf{C} = \sqrt{\mathbf{C}}\sqrt{\mathbf{C}} = \mathbf{UU}$$

with

$$\mathbf{U} = \sum_i \lambda_i \left(\boldsymbol{N}_i \boldsymbol{N}_i^T\right)$$

Let us define the tensor **R** as

$$\mathbf{R} = \mathbf{FU}^{-1}$$

We can then prove that **R** is a rotation tensor

$$\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-T}\mathbf{F}^T\mathbf{FU}^{-1} = \mathbf{U}^{-T}\mathbf{CU}^{-1} = \mathbf{U}^{-T}\mathbf{UUU}^{-1} = \mathbf{I}$$

The Polar Decomposition of Deformation Gradient

The new rotation tensor $\mathbf{R}$ implies that $\mathbf{F}$ can be decomposed as

$$\mathbf{F} = \mathbf{R}\mathbf{U}$$

The implication is

$$d\boldsymbol{x} = \mathbf{F}\boldsymbol{d}X = \mathbf{R}\left(\mathbf{U}d\boldsymbol{X}\right)$$

where $(\mathbf{U}d\boldsymbol{X})$ is a stretch along the eigenvectors and $\mathbf{R}$ is a rotation of the stretched vector.

Therefore

- $\mathbf{U}$ is called the stretch tensor
- $\mathbf{R}$ is called the rotation tensor

The Left Cauchy–Green Deformation Tensor

Next we will rewrite $d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2$ in terms of spatial coordinates,

$$\begin{aligned} d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2 &= d\boldsymbol{X}_1^T d\boldsymbol{X}_2 \\ &= \left(\mathbf{F}^{-1} d\boldsymbol{x}_1\right)^T \left(\mathbf{F}^{-1} d\boldsymbol{x}_2\right) \\ &= d\boldsymbol{x}_1^T \left(\underbrace{\mathbf{F}\mathbf{F}^T}_{\mathbf{B}}\right)^{-1} d\boldsymbol{x}_2 \end{aligned}$$

where

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T$$

is called the left Cauchy–Green Deformation tensor.

The Lagrange–Green Strain Tensor

We look at the deformation change. The difference between spatial and material dot products expressed in material coordinates,

$$\begin{aligned}\frac{1}{2}\left(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 - d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2\right) &= \frac{1}{2}\left(d\boldsymbol{x}_1^T d\boldsymbol{x}_2 - d\boldsymbol{X}_1^T d\boldsymbol{X}_2\right) \\ &= \frac{1}{2}\left(\left(\mathbf{F} d\boldsymbol{X}_1\right)^T \left(\mathbf{F} d\boldsymbol{X}_2\right) - d\boldsymbol{X}_1^T d\boldsymbol{X}_2\right) \\ &= d\boldsymbol{X}_1^T \underbrace{\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)}_{\mathbf{E}} d\boldsymbol{X}_2\end{aligned}$$

where

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$$

is called the Green Strain tensor.

The Euler–Almansi Strain Tensor

We describe the deformation change in terms of spatial coordinates,

$$
\begin{aligned}
\frac{1}{2}\left(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 - d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2\right) &= \frac{1}{2}\left(d\boldsymbol{x}_1^T d\boldsymbol{x}_2 - d\boldsymbol{X}_1^T d\boldsymbol{X}_2\right) \\
&= \frac{1}{2}\left(d\boldsymbol{x}_1^T d\boldsymbol{x}_2 - \left(\mathbf{F}^{-1} d\boldsymbol{x}_1\right)^T \left(\mathbf{F}^{-1} d\boldsymbol{x}_2\right)\right) \\
&= d\boldsymbol{x}_1^T \underbrace{\frac{1}{2}\left(\mathbf{I} - \left(\mathbf{F}\mathbf{F}^T\right)^{-1}\right)}_{\mathbf{e}} d\boldsymbol{x}_2
\end{aligned}
$$

where

$$
\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \left(\mathbf{F}\mathbf{F}^T\right)^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right)
$$

is called the Euler Strain tensor.

Spatial to Material Tensor Conversions

Observe that

$$\mathbf{e} = \mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1}$$
$$\mathbf{E} = \mathbf{F}^{T}\mathbf{e}\mathbf{F}$$

These are general tensor conversion formulas.

The Displacement Field

Let us define the displacement field as follows

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$$

Then we trivially have

$$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{X}$$

and

$$\mathbf{F} = \frac{\partial(\boldsymbol{u} + \boldsymbol{X})}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \mathbf{I}$$

or

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \mathbf{F} - \mathbf{I}$$

Putting Displacement Field into Play

Let us use $\mathbf{F} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \mathbf{I}$ in the definition of the Green Strain Tensor

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \mathbf{I}\right)^T \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \mathbf{I}\right) - \mathbf{I}\right) \\
&= \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) + \left(\mathbf{I}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T \mathbf{I}\right) + \left(\mathbf{I}^T \mathbf{I}\right) - \mathbf{I}\right) \\
&= \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T\right)
\end{aligned}
$$

Assuming Small Displacement Gradients

If $\| \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \| \ll 1$ then $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \approx \mathbf{0}$ and we have

$$\mathbf{E} \approx \varepsilon_0 = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)$$

This strain tensor is called the Cauchy Strain tensor. If we used $\mathbf{e}$ instead of $\mathbf{E}$ we would have found

$$\mathbf{e} \approx \varepsilon = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)$$

However, for small displacement gradients $x \approx X$, $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \approx \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$, and so $\varepsilon_0 \approx \varepsilon$.

The Strain Tensors

Green Strain Tensor

$$\mathbf{E} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T\right)$$

Cauchy Strain Tensor

$$\varepsilon \approx \varepsilon_0 = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)$$

Observe that using $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \mathbf{F} - \mathbf{I}$ we have

$$\varepsilon_0 = \frac{1}{2}\left(\mathbf{F}^T + \mathbf{F}\right) - \mathbf{I}$$

The Velocity

Since $\boldsymbol{x} = \Phi(\boldsymbol{X}, t)$ then the spatial velocity of a material point is

$$\boldsymbol{v}(\boldsymbol{X}, t) \equiv \dot{\boldsymbol{x}}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}(\boldsymbol{X}, t)}{\partial t} = \frac{\partial \Phi\left(\boldsymbol{X}, t\right)}{\partial t}$$

or given in terms of spatial coordinates

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}\left(\Phi^{-1}\left(\boldsymbol{x}, t\right), t\right)$$

The Velocity Gradient

The velocity gradient wrt. spatial coordinates is then

$$\mathbf{V} \equiv \frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}$$

Observe (Notice $\frac{\partial \boldsymbol{X}}{\partial t} = 0$)

$$\dot{\mathbf{F}} = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \boldsymbol{X}}\right) = \frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial \Phi}{\partial t}\right) = \frac{\partial}{\partial \boldsymbol{X}}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{X}} = \mathbf{V}\mathbf{F}$$

Resulting in

$$\mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

The Rate of Deformation

Taking the derivative

$$\frac{d}{dt}\left(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2\right) = d\boldsymbol{X}_1^T \dot{\mathbf{C}} d\boldsymbol{X}_2 = 2d\boldsymbol{X}_1^T \dot{\mathbf{E}} d\boldsymbol{X}_2$$

because $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$. The material strain rate tensor is

$$\dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \frac{1}{2}\left(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}\right)$$

Using $d\boldsymbol{X} = \mathbf{F}^{-1}d\boldsymbol{x}$ we have

$$\frac{1}{2}\frac{d}{dt}\left(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2\right) = d\boldsymbol{x}_1^T \underbrace{\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}}_{\mathbf{D}} d\boldsymbol{x}_2$$

where $\mathbf{D}$ is the rate of deformation tensor

Coordinate Conversion of Rate Tensors

Observe that similar to the Euler–Almansi and Lagrange–Green strain tensors we have

$$\mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$$
$$\dot{\mathbf{E}} = \mathbf{F}^{T}\mathbf{D}\mathbf{F}$$

From this, we also find that

$$\begin{aligned}\mathbf{D} &= \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1} \\ &= \mathbf{F}^{-T}\frac{1}{2}\left(\dot{\mathbf{F}}^{T}\mathbf{F} + \mathbf{F}^{T}\dot{\mathbf{F}}\right)\mathbf{F}^{-1} \\ &= \frac{1}{2}\left(\mathbf{V} + \mathbf{V}^{T}\right)\end{aligned}$$

because $\mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Thus $\mathbf{D}$ is the symmetric part of the velocity gradient.

Dynamics – Learning The Definitions of The Stress Tensors and Power

A Spatial Surface Element

The spatial surface element $d\boldsymbol{s}$ can be written as

$$d\boldsymbol{s} = \begin{bmatrix} d\boldsymbol{s}_1 \\ d\boldsymbol{s}_2 \\ d\boldsymbol{s}_3 \end{bmatrix}$$

where $d\boldsymbol{s}_i$ is the projection of the area of surface $d\boldsymbol{s}$ onto the i^{th} coordinate axis.

Observe that the surface normal is given by

$$\boldsymbol{n} = \frac{d\boldsymbol{s}}{\| d\boldsymbol{s} \|}$$

Cauchy's Stress Hypothesis

Definition: Stress is force per unit area. If we apply the force $d\boldsymbol{f}$ to the surface element $d\boldsymbol{s}$

$$
\begin{aligned}
d\boldsymbol{f}_1 &= \sigma_{11} d\boldsymbol{s}_1 + \sigma_{12} d\boldsymbol{s}_3 + \sigma_{13} d\boldsymbol{s}_3 \\
d\boldsymbol{f}_2 &= \sigma_{21} d\boldsymbol{s}_1 + \sigma_{22} d\boldsymbol{s}_3 + \sigma_{23} d\boldsymbol{s}_3 \\
d\boldsymbol{f}_3 &= \sigma_{31} d\boldsymbol{s}_1 + \sigma_{32} d\boldsymbol{s}_3 + \sigma_{33} d\boldsymbol{s}_3
\end{aligned}
$$

where σ_{ij} depends on position and time, collecting them

$$\sigma = \left[\sigma_{ij}\right]$$

we have

$$d\boldsymbol{f} = \sigma d\boldsymbol{s}$$

where σ is the Cauchy Stress Tensor.

The First Piola–Kirchhoff Stress Tensor

By definition the Cauchy stress tensor

$$d\boldsymbol{f} = \sigma d\boldsymbol{s}$$

The first Piola–Kirchhoff stess tensor is defined as

$$d\boldsymbol{f} = \mathbf{P} d\boldsymbol{S}$$

So we must have

$$\sigma d\boldsymbol{s} = d\boldsymbol{f} = \mathbf{P} d\boldsymbol{S}$$

Using Nanson's Relation

$$\sigma j \mathbf{F}^{-T} d\boldsymbol{S} = d\boldsymbol{f} = \mathbf{P} d\boldsymbol{S}$$

From this we have the relation between the Cauchy stress tensor and the first Piola–Kirchhoff stress tensor

The Second Piola–Kirchhoff Stress Tensor

By definition the Cauchy stress tensor

$$d\boldsymbol{f} = \sigma d\boldsymbol{s}$$

The second Piola–Kirchhoff stess tensor is defined as

$$d\boldsymbol{F} = \mathbf{S} d\boldsymbol{S}$$

However $d\boldsymbol{f} = \mathbf{F} d\boldsymbol{F}$ so we must have

$$\sigma d\boldsymbol{s} = d\boldsymbol{f} = \mathbf{F} d\boldsymbol{F} = \mathbf{F}\mathbf{S} d\boldsymbol{S}$$

$$\sigma d\boldsymbol{s} = \mathbf{F}\mathbf{S} d\boldsymbol{S}$$

$$j\mathbf{F}^{-1}\sigma\mathbf{F}^{-T} d\boldsymbol{S} = \mathbf{S} d\boldsymbol{S}$$

So

The Traction Force

We have

$$d\boldsymbol{f} = \sigma d\boldsymbol{s}$$

If we divide by surface area we will have traction $\boldsymbol{t} = d\boldsymbol{f} / \parallel d\boldsymbol{s} \parallel$

$$\boldsymbol{t} = \sigma \boldsymbol{n}$$

where

$$\begin{bmatrix} \boldsymbol{t}_1 \\ \boldsymbol{t}_2 \\ \boldsymbol{t}_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix}$$

Integration of Forces

Given the body volume force density $\boldsymbol{b}$ and arbitary spatial volume element v with spatial surface s then the total force on the volume is found by intergration

$$
\begin{aligned}
\mathcal{F} &= \int_v \boldsymbol{b}\, dv + \oint_s \boldsymbol{t}\, ds \\
&= \int_v \boldsymbol{b}\, dv + \oint_s (\sigma \boldsymbol{n})\, ds
\end{aligned}
$$

Using Volume Integrals

Recall Gauss–Divergence Theorem, for tensor field **A** over any closed volume v with surface s

$$\int_v \nabla \cdot \mathbf{A} dv = \oint_s \mathbf{A}\boldsymbol{n} ds$$

Applying this to

$$\mathcal{F} = \int_v \boldsymbol{b} dv + \oint_s \sigma \boldsymbol{n} ds$$

yields

$$\mathcal{F} = \int_v \underbrace{(\boldsymbol{b} + \nabla \cdot \sigma)}_{\boldsymbol{f}^*} dv$$

The Total Force and the Effective Force

In mechanical equilibrium, the effective force $\boldsymbol{f}^*$ must be zero since v was chosen arbitrarily. Thus writing it out

$$\boldsymbol{b} + \nabla \cdot \sigma = 0$$

This is Cauchy's Equation of Equilibrium. In non equilibrium, we must take inertia forces into account, $\mathcal{F} = \int_v \rho \ddot{\boldsymbol{x}} dv$,

$$\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \nabla \cdot \sigma$$

This is the motion of the equation in spatial coordinates for any continuum.

Recap Work and Power Definitions

Let $\boldsymbol{x}(t)$ be a particle and $\boldsymbol{f}$ some force acting on the particle. Work is force times distance, for a constant force over a displacement $\boldsymbol{u}$

$$W = \boldsymbol{f} \cdot \boldsymbol{u}$$

Or more general

$$W = \int \boldsymbol{f} \cdot d\boldsymbol{x}$$

Power is the rate at which work is performed

$$P \equiv \frac{dW}{dt}$$

The instantaneous power is $P = \boldsymbol{f} \cdot \boldsymbol{v}$ and so $W = \int_0^t \boldsymbol{f} \cdot \boldsymbol{v} dt$.

Power of The Effective Force

The power from the effective force

$$p = \boldsymbol{f}^* \cdot \boldsymbol{v}$$

The total power applied by the effective force

$$P = \int_v p dv = \int_v \boldsymbol{f}^* \cdot \boldsymbol{v} dv$$

Rewriting into

$$\begin{aligned} P &= \int_v (\boldsymbol{b} + (\nabla \cdot \sigma)) \cdot \boldsymbol{v} dv \\ &= \int_v (\nabla \cdot \sigma) \cdot \boldsymbol{v} dv + \int_v \boldsymbol{b} \cdot \boldsymbol{v} dv \end{aligned}$$

Using the "Product" Rule

Using

$$\nabla \cdot (\sigma \boldsymbol{v}) = (\nabla \cdot \sigma) \cdot \boldsymbol{v} + \sigma : (\nabla \boldsymbol{v}^T)$$

Then

$$P = \int_V (\nabla \cdot \sigma) \cdot \boldsymbol{v} dv + \int_V \boldsymbol{b} \cdot \boldsymbol{v} dv$$

Becomes

$$P = \int_V \nabla \cdot (\sigma \boldsymbol{v}) dv - \int_V \sigma : \mathbf{V}^T dv + \int_V \boldsymbol{b} \cdot \boldsymbol{v} dv$$

Recall $\nabla \boldsymbol{v}^T = \mathbf{V}^T$.

Apply Gauss–Divergence Theorem

Using Gauss–Divergence Theorem

$$P = \oint_s \boldsymbol{n} \cdot (\sigma \boldsymbol{v}) ds - \int_v \sigma : \mathbf{V}^T dv + \int_v \boldsymbol{b} \cdot \boldsymbol{v} dv$$

Using $\boldsymbol{t} = \sigma \boldsymbol{n}$ and $\sigma^T = \sigma$

$$P = \oint_s \boldsymbol{t} \cdot \boldsymbol{v} ds - \int_v \sigma : \mathbf{V} dv + \int_v \boldsymbol{b} \cdot \boldsymbol{v} dv$$

Using $\mathbf{D} = \frac{1}{2}\left(\mathbf{V} + \mathbf{V}^T\right)$ and symmetry of σ

$$P = \oint_s \boldsymbol{t} \cdot \boldsymbol{v} ds - \int_v \sigma : \mathbf{D} dv + \int_v \boldsymbol{b} \cdot \boldsymbol{v} dv$$

The Power in Spatial Coordinates

The internal stress power term in spatial coordinates

$$P_{\mathrm{e}} = \int_{v} \sigma : \mathbf{V} dv = \int_{v} \sigma : \mathbf{D} dv$$

Thus, in terms of power, we say that σ is work conjugate to $\mathbf{D}$. Our next task is to rewrite the internal power in terms of material coordinates. The result is to find the work conjugate quantities of

- The First Piola–Kirchhoff Stress tensor
- The Second Piola–Kirchhoff Stress tensor

Power Conjugency of $\mathbf{P}$

$$
\begin{aligned}
P_{\mathrm{e}} &= \int_v \sigma : \mathbf{V} dv && \text{(by definition)} \\
&= \int_V j\sigma : \mathbf{V} dV && (\text{by } dv = jdV) \\
&= \int_V j\sigma : \left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) dV && (\text{by } \mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1}) \\
&= \int_V \operatorname{tr}\left(j\sigma\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)\right) dV && (\text{by } \mathbf{A} : \mathbf{B} = \operatorname{tr}\left(\mathbf{A}^T\mathbf{B}\right)) \\
&= \int_V \operatorname{tr}\left(\left(j\mathbf{F}^{-1}\sigma\right)\dot{\mathbf{F}}\right) dV && (\operatorname{tr}\left(\mathbf{AB}\right) = \operatorname{tr}\left(\mathbf{BA}\right)) \\
&= \int_V \mathbf{P} : \dot{\mathbf{F}} dV && (\text{by } \operatorname{tr}\left(\mathbf{A}^T\mathbf{B}\right) = \mathbf{A} : \mathbf{B})
\end{aligned}
$$

Power Conjugency of $\mathbf{S}$

$$
\begin{aligned}
P_e &= \int_v \sigma : \mathbf{D} dv && \text{(by definition)} \\
&= \int_V j\sigma : \mathbf{D} dV && \text{(by } dv = jdV\text{)} \\
&= \int_V j\sigma : \left(\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}\right) dV && \text{(by } \mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}\text{)} \\
&= \int_V \operatorname{tr}\left(j\sigma\left(\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}\right)\right) dV && \text{(by } \mathbf{A} : \mathbf{B} = \operatorname{tr}\left(\mathbf{A}^T\mathbf{B}\right)\text{)} \\
&= \int_V \operatorname{tr}\left(\left(j\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}\right)\dot{\mathbf{E}}\right) dV && \text{(by } \operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})\text{)} \\
&= \int_V \mathbf{S} : \dot{\mathbf{E}} dV && \text{(by } \operatorname{tr}\left(\mathbf{A}^T\mathbf{B}\right) = \mathbf{A} : \mathbf{B}\text{)}
\end{aligned}
$$

The Energy Balance

For the equilibrium system mechanical energy balance means that $P = 0$. That is the time derivative of mechanical energy is zero – constant in time.
In the case of a non-equilibrium system we must include inertia forces,

$$P = \int_{V} (\rho \ddot{\mathbf{x}} - \boldsymbol{b} - \nabla \cdot \sigma) \cdot \boldsymbol{v} dv$$

Repeating the previous derivations to the total power becomes

$$P = \int_{V} \rho \ddot{\mathbf{x}} \cdot v dv + \int_{V} \sigma : \mathbf{D} dv - \oint_{S} \boldsymbol{t} \cdot \boldsymbol{v} dv - \int_{V} \boldsymbol{b} \cdot \boldsymbol{v} dv$$

and it must be zero for the conservation of mechanical energy.

Constitutive Equations

The Strain Energy in terms of $\mathbf{P}$

The total strain energy Ψ (The time integral of the "elastic" power P_e),

$$\Psi(\mathbf{F}) = \int_0^t \underbrace{\mathbf{P} : \dot{\mathbf{F}}}_{=\dot{\Psi}} dt$$

By the chain rule, we have

$$\dot{\Psi} = \frac{d}{dt}\Psi(\mathbf{F}) = \frac{\partial \Psi}{\partial \mathbf{F}} : \dot{\mathbf{F}}$$

So

$$\mathbf{P} : \dot{\mathbf{F}} = \dot{\Psi} = \frac{\partial \Psi}{\partial \mathbf{F}} : \dot{\mathbf{F}}$$

and we must have

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}$$

The Strain Energy in terms of **S**

The total strain energy Ψ,

$$\Psi(\mathbf{F}) = \int_0^t \underbrace{\mathbf{S} : \dot{\mathbf{E}}}_{=\dot{\Psi}} \, dt$$

By the chain rule, we have

$$\dot{\Psi} = \frac{d}{dt}\Psi(\mathbf{E}) = \frac{\partial \Psi}{\partial \mathbf{E}} : \dot{\mathbf{E}}$$

So

$$\mathbf{S} : \dot{\mathbf{E}} = \dot{\Psi} = \frac{\partial \Psi}{\partial \mathbf{E}} : \dot{\mathbf{E}}$$

and we must have

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}}$$

Isotropic Materials

The potential function is a function of invariants of **C**

$$\Psi(\mathbf{C}) = \Psi(\mathsf{I}_C, \mathsf{II}_C, \mathsf{III}_C)$$

The Second Piola–Kirchhoff Stress Tensor computed as

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathsf{I}_C}\mathbf{I} + 4\frac{\partial \Psi}{\partial \mathsf{II}_C}\mathbf{C} + 2j^2\frac{\partial \Psi}{\partial \mathsf{III}_C}\mathbf{C}^{-1}$$

The Saint Venant-Kirchhoff (SVK) Material Model

The simplest hyperelastic material model. The strain-energy is

$$\Psi(\mathbf{E}) = \frac{\lambda}{2}\left(\mathrm{tr}\left(\mathbf{E}\right)\right)^2 + \mu \mathrm{tr}\left(\mathbf{E}^2\right)$$

where λ and μ are the Lamé constants.

From $\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}}$ we have the stress-strain relation

$$\mathbf{S} = \lambda \mathrm{tr}\left(\mathbf{E}\right)\mathbf{I} + 2\mu\mathbf{E}$$

The **S**-**C** Relation using the SVK-Material

By definition $2\mathbf{E} + \mathbf{I} = \mathbf{C}$

$$\mathbf{S} = \frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \frac{\partial \mathbf{C}}{\partial \mathbf{E}} = 2 \frac{\partial \Psi}{\partial \mathbf{C}}$$

For isotropic materials stretch is independent of rotation $\Psi(\mathbf{C}) = \Psi(\mathsf{I}_C, \mathsf{II}_C, \mathsf{III}_C)$

$$\mathbf{S} = 2 \left(\frac{\partial \Psi}{\partial \mathsf{I}_C} \frac{\partial \mathsf{I}_C}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial \mathsf{II}_C} \frac{\partial \mathsf{II}_C}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial \mathsf{III}_C} \frac{\partial \mathsf{III}_C}{\partial \mathbf{C}} \right)$$

where

$$\frac{\partial \mathsf{I}_C}{\partial \mathbf{C}} = \mathbf{I}, \quad \frac{\partial \mathsf{II}_C}{\partial \mathbf{C}} = 2\mathbf{C}, \quad \text{and} \quad \frac{\partial \mathsf{III}_C}{\partial \mathbf{C}} = \det(\mathbf{C})\mathbf{C}^{-T} = j^2 \mathbf{C}^{-1}$$

Assignment

- Discuss why the invariants on Pages 8 and 9 are invariant (ie. independent of rotation).
- Derive the formulas on page 11 from the formulas given in terms of **C** on Page 8-9. If you have time derive an eigenvalue version of the invariant $\mathbf{II}_C^*$.
- On page 38 we exploit the symmetry of the Cauchy stress tensor. Now prove that if $\sigma = \sigma^T$ then $\sigma : \mathbf{V}^T = \sigma : \mathbf{V}$ and that $\sigma : \mathbf{V} = \sigma : \mathbf{D}$.
- Derive the formula for **S** on the page 46.
- Derive the stress-strain relation on the page 47 by differentiation of Ψ.

Assignment

Download `https://github.com/erleben/hyper-sim` run a bending rod simulation case and find a way to visualize the stress tensor field.

- Use the FEM method in the framework with the adaptive time stepping parameter on.
- Try and play around with the Young modulus and Poisson ratio values.

If you have time try to change the stress-strain relation used in the FVM or FEM methods to that of a neoHookean material (See Bonet and Wood text for examples on constitutive laws).