Directional Derivatives

A Tool for Linearization

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Directional Derivative of Deformation Gradient (1/3)

Let $\mathbf{u} \neq \mathbf{0}$ be some arbitrary direction

$$D\mathbf{F}(\mathbf{x})[\mathbf{u}] = \lim_{s \to 0} \frac{\mathbf{F}(\mathbf{x} + s\mathbf{u}) - \mathbf{F}(\mathbf{x})}{s}$$

Recall that $\mathbf{x} = \Phi(\mathbf{X})$ and $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \Phi(\mathbf{X})}{\partial \mathbf{X}}$ so

$$\mathbf{F}(\mathbf{x} + s\mathbf{u}) = \frac{\partial \Phi(\Phi^{-1}(\mathbf{x} + s\mathbf{u}))}{\partial \mathbf{X}} = \frac{\partial (\mathbf{x} + s\mathbf{u})}{\partial \mathbf{X}}$$

Directional Derivative of Deformation Gradient (2/3)

Then

$$
\begin{aligned}
D\mathbf{F}(\mathbf{x})[\mathbf{u}] &= \lim_{s\to 0} \frac{\frac{\partial(\mathbf{x}+s\mathbf{u})}{\partial \mathbf{X}} - \frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{X}}}{s} \\
&= \lim_{s\to 0} \frac{\frac{\partial \mathbf{x}}{\partial \mathbf{X}} + s\frac{\partial \mathbf{u}}{\partial \mathbf{X}} - \frac{\partial \mathbf{x}}{\partial \mathbf{X}}}{s} \\
&= \frac{\partial \mathbf{u}}{\partial \mathbf{X}}
\end{aligned}
$$

Directional Derivative of Deformation Gradient (3/3)

Using the chain rule we may rewrite

$$D\mathbf{F}(\mathbf{x})[\mathbf{u}] = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = (\nabla \mathbf{u})\mathbf{F}$$

Observe that if $\mathbf{u}$ is considered to be a function of $\mathbf{X}$ and not $\mathbf{x}$ then we write

$$\mathbf{u}_0 = \mathbf{u}(\mathbf{X})$$

and then we have

$$D\mathbf{F}(\mathbf{x})[\mathbf{u}_0] = \frac{\partial \mathbf{u}_0}{\partial \mathbf{X}} = \nabla_0 \mathbf{u}_0$$

Directional Derivative of Green Strain Tensor

By definition we have

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - I\right)$$

so

$$\begin{aligned} D\mathbf{E}(\mathbf{x})[\mathbf{u}] &= \frac{1}{2}\left(D\mathbf{F}(\mathbf{x})[\mathbf{u}]^T\mathbf{F} + \mathbf{F}^T D\mathbf{F}(\mathbf{x})[\mathbf{u}]\right) \\ &= \frac{1}{2}\mathbf{F}^T\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}^T + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)\mathbf{F} \\ &= \mathbf{F}^T\varepsilon\mathbf{F} \end{aligned}$$

Directional Derivative of 2nd Piola Kirchhoff Stress Tensor (1/3)

We know

$$D\mathbf{S}(\mathbf{x})[\mathbf{u}] = D\left(\mathbf{S}(\mathbf{E}(\mathbf{x}))\right)[\mathbf{u}]$$

From the chain rule we have

$$D\mathbf{S}_{IJ}(\mathbf{x})[\mathbf{u}] = \sum_{K=1}^{3}\sum_{L=1}^{3}\frac{\partial \mathbf{S}_{IJ}(\mathbf{x})}{\partial \mathbf{E}_{KL}} D\mathbf{E}_{KL}(\mathbf{x})[\mathbf{u}]$$

Directional Derivative of 2nd Piola Kirchhoff Stress Tensor (2/3)

That is

$$D\mathbf{S}(\mathbf{x})[\mathbf{u}] = \mathcal{C}(\mathbf{x}) : D\mathbf{E}(\mathbf{x})[\mathbf{u}]$$

where the fourth order tensor $\mathcal{C}$ is given as

$$\mathcal{C}(\mathbf{x}) = \frac{\partial \mathbf{S}(\mathbf{x})}{\partial \mathbf{E}}$$

Directional Derivative of 2nd Piola Kirchhoff Stress Tensor (3/3)

We know that the partial derivative of the strain energy gives us the stress tensor

$$\mathbf{S} = \frac{\partial \psi}{\partial \mathbf{E}} = 2\frac{\partial \psi}{\partial \mathbf{C}}$$

So that means

$$\mathcal{C} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial}{\partial \mathbf{E}}\left(2\frac{\partial \psi}{\partial \mathbf{C}}\right) = 2\frac{\partial^2 \psi}{\partial \mathbf{E} \partial \mathbf{C}} = 4\frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}}$$

Or in index notation

$$\mathcal{C}_{IJKL} = \frac{4\partial^2 \psi}{\partial \mathbf{C}_{IJ} \partial \mathbf{C}_{KL}} = \mathcal{C}_{KLIJ}$$