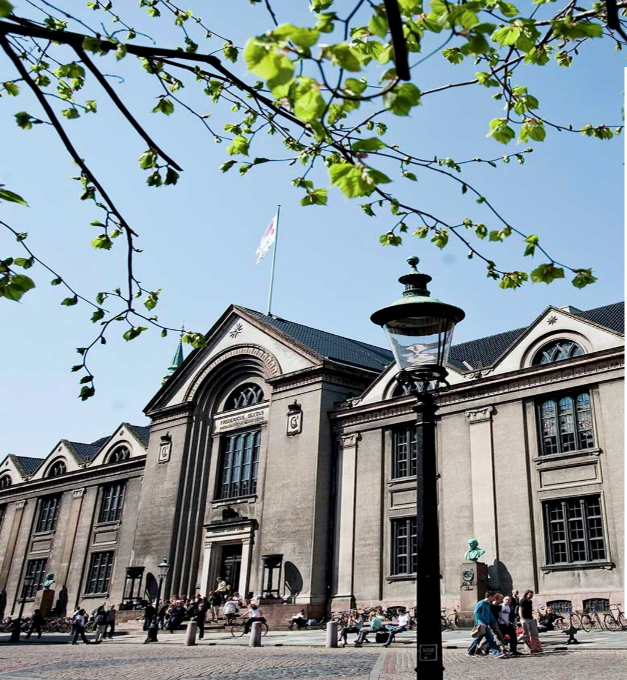

Contact Force Problems

Why we need Complementarity Problems

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The Position Level Non-Penetrating Constraint

Consider two rigid bodies

- Let d me the minimum distance
- Penetration is not allowed hence $d \geq 0$
- A normal force $\mathbf{f}_n$ prevents penetration and it is non-sticking $\mathbf{f}_n \geq 0$
- If the rigid bodies are separated $d > 0$ then there can be no normal force $\mathbf{f}_n = 0$ between the two bodies.
- If there is a normal force $\mathbf{f}_n > 0$ then the rigid bodies must be touching $d = 0$.

We write

$$\mathbf{f}_n > 0 \Rightarrow d = 0$$
$$d > 0 \Rightarrow \mathbf{f}_n = 0$$

This is known as a complementarity condition

The Velocity Level Non-Penetrating Constraint

Assume we have touching contact $d = 0$ and let $\mathbf{u} = \frac{d}{dt}\mathbf{d}$ is the time derivative of the minimum distance vector. Let $\mathbf{u}_n = \mathbf{n} \cdot \mathbf{u}$ be the projection onto the normal direction at the point of contact.

- if the contact is about to separate then $\mathbf{u}_n > 0$
- If we have sustained contact then $\mathbf{u}_n = 0$
- If $\mathbf{u}_n > 0$ then at any future time we have $d > 0$ due to continuity of physics, this implies $\mathbf{f}_n = 0$
- If $\mathbf{f}_n > 0$ then the contact can not be separated and we must have sustained contact $\mathbf{u}_n = 0$

We write

$$\mathbf{f}_n > 0 \Rightarrow \mathbf{u}_n = 0$$
$$\mathbf{u}_n > 0 \Rightarrow \mathbf{f}_n = 0$$

The Contact Frame

Let the contact normal be $\mathbf{n}$ and the orthogonal contact plane vector $\mathbf{t}$. We assume

$$\begin{aligned} \| \mathbf{n} \| &= 1 \\ \| \mathbf{t} \| &= 1 \\ \mathbf{n} \cdot \mathbf{t} &= 0 \end{aligned}$$

Define the contact plane velocities as

$$\begin{aligned} u_{t_1} &= \mathbf{t} \cdot \mathbf{u} \\ u_{t_2} &= -\mathbf{t} \cdot \mathbf{u} \end{aligned}$$

and $\mathbf{u}_n = \mathbf{n} \cdot \mathbf{u}$.

The Friction Force

- The friction force $\mathbf{f}_t$ is given by

$$\mathbf{f}_t = \lambda_1 \mathbf{t} - \lambda_2 \mathbf{t}$$

where

$$\lambda_1 > 0 \Rightarrow \lambda_2 = 0$$
$$\lambda_2 > 0 \Rightarrow \lambda_1 = 0$$

The λ's are the coefficients of the friction force projected onto the positive span defined by $\mathbf{t}$ and $-\mathbf{t}$.

Coulomb Friction in a 2D World

- If there is a normal force $\mathbf{f}_n$ then the friction force $\mathbf{f}_t$ is bounded by the cone

$$\| \mathbf{f}_t \| = \lambda_1 + \lambda_2 \leq \mu \mathbf{f}_n$$

  where $\mu > 0$ is a positive constant known as the coefficient of friction.
- If there is sliding $\mathbf{u}_t \neq \mathbf{0}$ then the friction force $\mathbf{f}_t$ works against the sliding direction and attains it maximum possible value.
- If there is no sliding then the friction force can have any value bounded by the friction cone.

Coulomb Friction in a 2D World Continued

Let us introduce the scalar $\beta \geq 0$ then we may write

$$
\begin{aligned}
\lambda_1 > 0 &\Rightarrow (\beta - u_{t_1}) = 0 \\
\lambda_2 > 0 &\Rightarrow (\beta - u_{t_2}) = 0 \\
(\beta - u_{t_1}) > 0 &\Rightarrow \lambda_1 = 0 \\
(\beta - u_{t_2}) > 0 &\Rightarrow \lambda_2 = 0
\end{aligned}
$$

Hence

- if there is sliding (a λ is positive) then β estimates the magnitude of the sliding speed

Now we combine with

$$
\begin{aligned}
\beta > 0 &\Rightarrow (\mu \mathbf{f}_n - \lambda_1 - \lambda_2) = 0 \\
(\mu \mathbf{f}_n - \lambda_1 - \lambda_2) > 0 &\Rightarrow \beta = 0
\end{aligned}
$$