The Fluid Problem

A Short Introduction to Navier Stokes Equations

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The Incompressible Navier Stokes Equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = \mu\nabla^2\mathbf{u} - \nabla p + \mathbf{F}$$

where

- Position by $\mathbf{x} \in \mathbb{R}^3$ with SI units(^{1}) $[m^n]$, time by $t \in \mathbb{R}$ with $[s]$, and μ is a material coefficient, dynamic viscosity, with $\left[Ns/m^2\right]$.
- The density field - mass per volume - by $\rho(\mathbf{x}, t) \in \mathbb{R}$ with $\left[Kg/m^3\right]$
- The velocity field is given by $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ with $\left[m/s/m^3\right]$
- The pressure field - force per area per volume - by $p(\mathbf{x}, t) \in \mathbb{R}$ with $\left[N/m^2/m^3\right]$
- The external force field is given by $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$ with $\left[N/m^3\right]$

The Nabla Operator

So ∇ means

$$\nabla \equiv \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Hence ∇p means

$$\nabla p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix}$$

Examples of using the Nabla Operator

$\nabla^2 \mathbf{u}$ means

$$\nabla^2 \mathbf{u} = (\nabla \cdot \nabla)\mathbf{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{u}$$

and given $\mathbf{u} = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \end{bmatrix}^T$ we have

$$\nabla^2 \mathbf{u} = \begin{bmatrix} \frac{\partial^2 \mathbf{u}_x}{\partial x^2} + \frac{\partial^2 \mathbf{u}_x}{\partial y^2} + \frac{\partial^2 \mathbf{u}_x}{\partial z^2} \\ \frac{\partial^2 \mathbf{u}_y}{\partial x^2} + \frac{\partial^2 \mathbf{u}_y}{\partial y^2} + \frac{\partial^2 \mathbf{u}_y}{\partial z^2} \\ \frac{\partial^2 \mathbf{u}_z}{\partial x^2} + \frac{\partial^2 \mathbf{u}_z}{\partial y^2} + \frac{\partial^2 \mathbf{u}_z}{\partial z^2} \end{bmatrix}$$

Intuition About the Navier Stokes Terms

$$\underbrace{\rho\left(\overbrace{\frac{\partial \mathbf{u}}{\partial t}}^{\text{Unsteady}} + \overbrace{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}^{\text{Advection}}\right)}_{\text{Inertia}} = \underbrace{\mu\nabla^2\mathbf{u}}_{\text{Diffusion}} \underbrace{-\nabla p}_{\text{Pressure forces}} \underbrace{+\mathbf{F}}_{\text{External forces}}$$

Question: What is the difference between advection and convection?

More Technical Explanation

Taken from Wikipedia

> The term advection sometimes serves as a synonym for convection, but technically, convection covers the sum of transport by diffusion and by advection. Advective transport describes the movement of some quantity via the bulk flow of a fluid (as in a river or pipeline).

The Incompressible Navier Stokes Equations Again

Question: How many unknowns do we have and how many equations do we have?

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right) = \mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{F}$$

Assume ρ, μ and $\mathbf{F}$ are constants.

Oh no!

Conclusion we are missing one equation

Let us try and fix this.

The Continuity Equation

From the conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Usually, we assume the density field to be a material constant ($\frac{\partial \rho}{\partial t} = 0$ and $\nabla \rho = \mathbf{0}$) hence

$$\nabla \cdot (\rho \mathbf{u}) = 0$$

Under these assumptions, we have simply

$$\nabla \cdot \mathbf{u} = 0$$

This condition is sometimes termed the divergence-free condition or the no volume loss condition.

The Incompressible Navier Stokes Equations

Finally, we have the version of the incompressible Navier Stokes equations we will be studying in this course,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = \mu\nabla^2\mathbf{u} - \nabla p + \mathbf{F}$$

$$\nabla\cdot\mathbf{u} = 0$$

These equations “describe” how fluid moves inside a domain.

Question: What happens when a fluid touches a solid wall?

Boundary Conditions for Vacuum (1/2)

Assuming no matter outside the fluid surface the pressure must be zero,

$$p_{\text{vacuum}} = 0$$

Since we have no matter outside the fluid there is nothing that can move around there. Hence we may think we need

$$\mathbf{v}_{\text{vacuum}} = \mathbf{0}\,.$$

Boundary Conditions for Vacuum (2/2)

A different view: Vacuum means there is nothing to resist the fluid in moving hence the vacuum must have the same velocity as the fluid,

$$\Delta \mathbf{v} = \mathbf{0} \quad \mathbf{x} \in \Gamma_{\text{vacuum}}$$

This is a benefit as we otherwise would numerically have a jump in velocity across the fluid-vacuum interface to deal with. Using the different view we have a "smooth" field everywhere.

Solid Wall Boundary Conditions (1/2

At the solid wall, the fluid will at a “microscopic” level “stick” to the wall and move with the wall.

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{\text{wall}}, \quad \mathbf{x} \in \Gamma_{\text{wall}}$$

where Γ_{wall} is the point set of the wall boundary and $\mathbf{u}_{\text{wall}}$ the velocity field of the wall.

Usually in simulations walls are static meaning,

$$\mathbf{u}_{\text{wall}} = 0$$

Solid Wall Boundary Conditions (2/2)

The difference of the pressure field across the solid wall boundary in the normal direction is zero, this we write as

$$\frac{\partial p}{\partial \mathbf{n}} = 0$$

where $\mathbf{n}$ is the unit outward normal of the solid wall surface. The term on the left is the directional derivative in the normal direction. It says the slope of p in the direction of $\mathbf{n}$ is zero.

A Simple Solid Wall Boundary Conditions

For some applications, it makes sense to allow the fluid to slip along the tangential direction of the solid wall and only stick in the normal direction. This means

$$\mathbf{n} \cdot \mathbf{u}(\mathbf{x}, t) = \mathbf{n} \cdot \mathbf{u}_{\text{wall}}, \quad \mathbf{x} \in \Gamma_{\text{wall}}$$

When assuming static solid walls we have

$$\mathbf{n} \cdot \mathbf{u} = 0$$

Question: Think about what boundary conditions to have on $\mathbf{u}$ and p on a free surface (like a water surface).

Force Equilibrium at Free Surface Interface

Surface tension forces are given by

$$\mathbf{f}(\mathbf{x}) = -\gamma \nabla \mathcal{A}(\mathbf{x})$$

where $\gamma \in \mathbb{R}^+$ is surface tension, a material constant, and $\mathcal{A}$ is the area of the interface.

Looking at a sufficiently small arbitrary area A of the interface between fluids a and b then force equilibrium dictates that all forces add to zero

$$\int_A p_a \mathbf{n} dA - \int_A p_b \mathbf{n} dA - \int_A \gamma \nabla \mathcal{A} dA = \mathbf{0}$$

where $\mathbf{n}$ is the unit normal of the interface pointing from a to b.d

The Young -Laplace Equation

Now

$$\int_A \left((p_a - p_b)\,\mathbf{n} - \gamma \nabla \cdot \mathbf{n} \right) dA = 0$$

Resulting in the Young-Laplace equation

$$(p_a - p_b) = -\gamma \nabla \cdot \mathbf{n}$$

From differential geometry, we have $-\nabla \cdot \mathbf{n} = 2\,H$ where H is the mean curvature.

Boundary Conditions for Free Surfaces (1/2)

The Young-Laplace equation gives a boundary condition on the pressure field for a fluid in equilibrium,

$$\frac{\partial p}{\partial n} = \nabla p \cdot n = (p_a - p_b) = \gamma \nabla \cdot \mathbf{n} = \gamma \, 2 \, H$$

stating that the directional derivative of the pressure field in the normal direction is linearly proportional to the mean curvature H. This means the pressure field can be discontinuous across the normal direction at the free surface interface.

Boundary Conditions for Free Surfaces (2/2)

For fluid phases, the velocity field is assumed to "stick" at the interface they meet this means no separation or sliding motion,

$$\mathbf{u}_a = \mathbf{u}_b$$

Notice the velocity field across the interface is continuous but could be non-smooth. That is we can have a jump discontinuity in velocity gradients.

First Approach

Deriving Navier Stokes from Conservation Principles

Reynolds Transport Theorem

Given a volume V with surface S and some quantity ϕ (scalar, vector or tensor) then

$$\frac{d}{dt}\int_V \phi dV = \int_V \frac{\partial \phi}{\partial t} dV + \oint_S \phi \left(\mathbf{n} \cdot (\mathbf{u} - \mathbf{v})\right) dS$$

where $\mathbf{n}$ is outward unit normal of the surface, $\mathbf{v}$ is the velocity of the surface and $\mathbf{u}$ is the velocity of the quantity ϕ.

- Eulerian representation the volume stays fixed in world $\mathbf{v} = \mathbf{0}$
- Lagrangian representation the volume follows the quantity $\mathbf{v} = \mathbf{u}$

Conservation implies $\frac{d}{dt}\int_V \phi dV = 0$, hence we have

$$\int_V \frac{\partial \phi}{\partial t} dV + \oint_S \phi \left(\mathbf{n} \cdot \mathbf{u}\right) dS = 0$$

Conservation of Mass

We choose $\phi = \rho$ then we have

$$\int_V \frac{\partial \rho}{\partial t} dV + \oint_S \rho \left(\mathbf{n} \cdot \mathbf{u} \right) dS = 0$$

We use Gauss-Divergence theorem

$$\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u} \right) dV = 0$$

As V is arbitrary

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u} \right) = 0$$

Hence we have derived the continuity equation

Balance (Conservation) of Momentum (1/3)

We choose $\phi = \rho\mathbf{u}$ then we know

$$\frac{d}{dt}\int_V \rho\mathbf{u} dV = \sum \mathbf{f}$$

where $\mathbf{f}$ are the forces acting on the fluid volume V. Using Reynolds transport theorem we have

$$\int_V \frac{\partial \rho\mathbf{u}}{\partial t} dV + \oint_S \rho\mathbf{u}\left(\mathbf{n}\cdot\mathbf{u}\right) dS = \sum \mathbf{f}$$

The sum of forces is split into surface forces and the body forces

$$\sum \mathbf{f} = \mathbf{f}_{\text{surface}} + \mathbf{f}_{\text{body}}$$

Balance (Conservation) of Momentum (2/3)

So we have the sum of surface forces and the body forces

$$\sum \mathbf{f} = \mathbf{f}_{\text{surface}} + \mathbf{f}_{\text{body}}$$

The surface and body forces are given by

$$\mathbf{f}_{\text{surface}} = \oint_S \mathbf{T}\mathbf{n} dS$$
$$\mathbf{f}_{\text{body}} = \int_V \rho \mathbf{b} dV$$

Here body forces are given per unit mass. The first integral is Cauchy's stress hypothesis where $\mathbf{T}$ is the stress tensor.

Balance (Conservation) of Momentum (3/3)

So we have

$$\int_V \frac{\partial \rho \mathbf{u}}{\partial t} dV + \oint_S \rho \mathbf{u} \left(\mathbf{n} \cdot \mathbf{u} \right) dS = \oint_S \mathbf{T}\mathbf{n} dS + \int_V \rho \mathbf{b} dV$$

Using the Gauss-Divergence theorem and given V is arbitrary we have

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \nabla \right) \left(\rho \mathbf{u} \right) = \nabla \cdot T + \rho \mathbf{b}$$

We are almost done, all we need is a constitutive equation, also called an equation of state (EOS), that tells us how **T** is defined.

Newtonian Fluids – An Equation of State Example (1/2)

For Newtonian Fluids **T** is defined as follows

$$\mathbf{T} = -\left(p + \frac{2}{3}\mu \nabla \cdot \mathbf{u}\right)\mathbf{I} + 2\mu\mathbf{D}$$

where μ is dynamic viscosity, **I** the identity tensor, p is the pressure field and **D** is the rate of strain (deformation) tensor is given by

$$\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

The symmetric part of the gradient of the velocity field.

Newtonian Fluids – An Equation of State Example (2/2)

For incompressible fluids $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ substitution into the equation of state,

$$\mathbf{T} = -\left(p + \frac{2}{3}\mu \nabla \cdot \mathbf{u}\right)\mathbf{I} + 2\mu \mathbf{D}$$

We have the special case of the Newtonian Incompressible fluid stress tensor,

$$\mathbf{T} = -p\mathbf{I} + \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

Intuition

Newtonian fluids have no resistance to shear forces

$$\mathbf{T} = -\left(\underbrace{p}_{\text{resist compression}} + \underbrace{\frac{2}{3}\mu\nabla\cdot\mathbf{u}}_{\text{damping compression}} \right)\mathbf{I} + \underbrace{2\mu\mathbf{D}}_{\text{damping shearing}}$$

For incompressible fluids no damping of compression, only damping of shearing motion BUT no resistance.

Newtonian Fluids

By definition of ∇ (see lecture slides 14)

$$\mathbf{T} = -\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \mu \left(\begin{bmatrix} \frac{\partial \mathbf{u}_x}{\partial x} & \frac{\partial \mathbf{u}_x}{\partial y} & \frac{\partial \mathbf{u}_x}{\partial z} \\ \frac{\partial \mathbf{u}_y}{\partial x} & \frac{\partial \mathbf{u}_y}{\partial y} & \frac{\partial \mathbf{u}_y}{\partial z} \\ \frac{\partial \mathbf{u}_z}{\partial x} & \frac{\partial \mathbf{u}_z}{\partial y} & \frac{\partial \mathbf{u}_z}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{u}_x}{\partial x} & \frac{\partial \mathbf{u}_y}{\partial x} & \frac{\partial \mathbf{u}_z}{\partial x} \\ \frac{\partial \mathbf{u}_x}{\partial y} & \frac{\partial \mathbf{u}_y}{\partial y} & \frac{\partial \mathbf{u}_z}{\partial y} \\ \frac{\partial \mathbf{u}_x}{\partial z} & \frac{\partial \mathbf{u}_y}{\partial z} & \frac{\partial \mathbf{u}_z}{\partial z} \end{bmatrix} \right)$$

$$= -\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \mu \begin{bmatrix} \left(\frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \mathbf{u}_x}{\partial x}\right) & \left(\frac{\partial \mathbf{u}_x}{\partial y} + \frac{\partial \mathbf{u}_y}{\partial x}\right) & \left(\frac{\partial \mathbf{u}_x}{\partial z} + \frac{\partial \mathbf{u}_z}{\partial x}\right) \\ \left(\frac{\partial \mathbf{u}_y}{\partial x} + \frac{\partial \mathbf{u}_x}{\partial y}\right) & \left(\frac{\partial \mathbf{u}_y}{\partial y} + \frac{\partial \mathbf{u}_y}{\partial y}\right) & \left(\frac{\partial \mathbf{u}_y}{\partial z} + \frac{\partial \mathbf{u}_z}{\partial y}\right) \\ \left(\frac{\partial \mathbf{u}_z}{\partial x} + \frac{\partial \mathbf{u}_x}{\partial z}\right) & \left(\frac{\partial \mathbf{u}_z}{\partial y} + \frac{\partial \mathbf{u}_y}{\partial z}\right) & \left(\frac{\partial \mathbf{u}_z}{\partial z} + \frac{\partial \mathbf{u}_z}{\partial z}\right) \end{bmatrix}$$

Next we need to compute $\nabla \cdot \mathbf{T}$

Newtonian Fluids

So the first component of $\nabla \cdot \mathbf{T}$ becomes

$$(\nabla \cdot \mathbf{T})_x = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \mathbf{u}_x}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial \mathbf{u}_x}{\partial y} + \frac{\partial \mathbf{u}_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial \mathbf{u}_x}{\partial z} + \frac{\partial \mathbf{u}_z}{\partial x}\right)\right)$$

Newtonian Fluids

So the first component of $\nabla \cdot \mathbf{T}$ becomes

$$(\nabla \cdot \mathbf{T})_x = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial \mathbf{u}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial \mathbf{u}_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \mathbf{u}_x}{\partial z}\right)$$
$$+ \frac{\partial}{\partial x}\left(\mu \frac{\partial \mathbf{u}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial \mathbf{u}_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \mathbf{u}_z}{\partial x}\right)$$

Assume μ is constant

$$(\nabla \cdot \mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 \mathbf{u}_x}{\partial x^2} + \frac{\partial^2 \mathbf{u}_x}{\partial y^2} + \frac{\partial^2 \mathbf{u}_x}{\partial z^2}\right) + \mu \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \mathbf{u}_y}{\partial y} + \frac{\partial \mathbf{u}_z}{\partial z}\right)$$

Newtonian Fluids

For incompressible fluids we have $\nabla \cdot \mathbf{u} = 0$ so we find

$$(\nabla \cdot \mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 \mathbf{u}_x}{\partial x^2} + \frac{\partial^2 \mathbf{u}_x}{\partial y^2} + \frac{\partial^2 \mathbf{u}_x}{\partial z^2} \right) + \mu \frac{\partial}{\partial x} \left(\frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \mathbf{u}_y}{\partial y} + \frac{\partial \mathbf{u}_z}{\partial z} \right)$$

Becomes

$$(\nabla \cdot \mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu \nabla^2 \mathbf{u}_x$$

Similar for $(\nabla \cdot \mathbf{T})_y$ and $(\nabla \cdot \mathbf{T})_z$ so

$$\nabla \cdot \mathbf{T} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

The Incompressible Navier Stokes Equations

Assume incompressible fluids and constant viscosity the balance of momentum equations now read

$$\frac{\partial \rho \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)(\rho \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mu \equiv \text{const}$$

Second Approach

Deriving Navier Stokes from Limits Consideration

Deriving the Navier Stokes Equation From Limits (1/2)

Assume we have a sufficient small parcel of square fluid volume and consider conservation of momentum in x-direction.

$$
\begin{aligned}
\frac{D\left(m\,\mathbf{u}_x\right)}{Dt} &= \mathbf{f}_x = \Delta y\,\Delta z\,p_x^- - \Delta y\,\Delta z\,p_x^+ \\
m\,\frac{D\mathbf{u}_x}{Dt} &= \Delta y\,\Delta z\,p_x^- - \Delta y\,\Delta z\,p_x^+ \\
\rho\,\Delta x\,\Delta y\,\Delta z\,\frac{D\mathbf{u}_x}{Dt} &= \Delta y\,\Delta z\,p_x^- - \Delta y\,\Delta z\,p_x^+ \\
\rho\,\frac{D\mathbf{u}_x}{Dt} &= -\frac{p_x^+ - p_x^-}{\Delta x}
\end{aligned}
$$

Deriving the Navier Stokes Equation From Limits (2/2)

Taking the limit of infinitely small parcels of fluid we find,

$$\lim_{\Delta x \to 0} \rho \frac{D\mathbf{u}_x}{Dt} = \rho \frac{D\mathbf{u}_x}{Dt} = -\lim_{\Delta x \to 0} \frac{p_x^+ - p_x^-}{\Delta x} = -\frac{\partial p}{\partial x}$$

That is

$$\rho \frac{D\mathbf{u}_x}{Dt} = -\frac{\partial p}{\partial x}$$

Adding *y* and *z* directions too

So we have

$$\rho \frac{D\mathbf{u}_x}{Dt} = -\frac{\partial p}{\partial x}$$
$$\rho \frac{D\mathbf{u}_x}{Dt} = -\frac{\partial p}{\partial y}$$
$$\rho \frac{D\mathbf{u}_z}{Dt} = -\frac{\partial p}{\partial x}$$

Or more compactly written

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p$$

Observation: fluids flow from high pressure towards low pressure.

Adding other forces

Adding a body force such as gravity $m\mathbf{b}$ would result in

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \underbrace{\rho \mathbf{b}}_{\mathbf{f}}$$

Notice that $\mathbf{f}$ is a force density.

Viscous damping coud be added as $\nabla \cdot 2\mu\mathbf{D}$.

The inertia term

So given $\mathbf{u}(\mathbf{x}(t), t)$ then

$$\rho \frac{D\mathbf{u}(\mathbf{x}(t), t)}{Dt} = \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \left(\frac{\partial \mathbf{u}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{u}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{u}}{\partial z}\frac{\partial z}{\partial t} \right)$$
$$= \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \left(\mathbf{u} \cdot \nabla \right) \mathbf{u}$$

This assumes ρ is constant.

Putting it Together

The Inviscid Navier Stokes

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

Inviscid means zero viscosity.

Third Approach

Deriving Navier Stokes from Taylor Approximations

First Order Taylor is Accurate

Assume we are observing a small cube volume (Δx, Δy, and Δz) of a large fluid and the volume is sufficiently small such that first-order Taylor approximation of p, $\mathbf{u}$ and ρ are accurate.

First Order Taylor Approximations (1/2)

Pressure at the left and right wall centers are

$$p_x^- = p - \frac{\partial p}{\partial x}\frac{1}{2}\Delta x$$
$$p_x^+ = p + \frac{\partial p}{\partial x}\frac{1}{2}\Delta x$$

Velocities at the left and right wall centers

$$\mathbf{u}_x^- = \mathbf{u} - \frac{\partial \mathbf{u}}{\partial x}\frac{1}{2}\Delta x$$
$$\mathbf{u}_x^+ = \mathbf{u} + \frac{\partial \mathbf{u}}{\partial x}\frac{1}{2}\Delta x$$

First Order Taylor Approximations (2/2)

For mass densities we obtain,

$$\rho_x^- = \rho - \frac{\partial \rho}{\partial x}\frac{1}{2}\Delta x$$
$$\rho_x^+ = \rho + \frac{\partial \rho}{\partial x}\frac{1}{2}\Delta x$$

Same for up/down and front/back (y and z-direction) faces

The x-direction

Analyzing the x-direction we have from the conservation of momentum

$$\frac{D}{Dt}\left(\rho\,\Delta x\,\Delta y\,\Delta z\,\mathbf{u}_x\right) = p_x^-\,\Delta y\,\Delta z - p_x^+\,\Delta y\,\Delta z$$
$$= -\frac{\partial p}{\partial x}\,\Delta x\,\Delta y\,\Delta z$$

Hence we find

$$\frac{D}{Dt}\left(\rho\mathbf{u}_x\right) = -\frac{\partial p}{\partial x}$$

Adding y and z directions give the inviscid Navier-Stokes equations as before.

Conservation of Mass

Inside the volume we have

$$\frac{D}{Dt}\left(\rho\,\Delta x\,\Delta y\,\Delta z\right) = \frac{\partial \rho}{\partial t}\,\Delta x\,\Delta y\,\Delta z$$

The mass flux over a surface element dA with outward unit normal $\mathbf{n}$ is given by

$$M = \rho\mathbf{u}\cdot\mathbf{n}dA$$

Hence the flow in/out of the left/right side faces of the volume become

$$\begin{aligned} M_x^- &= \Delta y\,\Delta z\,\rho_x^+\,\mathbf{u}_x^- \\ M_x^+ &= -\Delta y\,\Delta z\,\rho_x^-\,\mathbf{u}_x^+ \end{aligned}$$

Conservation of Mass

Conservation of mass means the inner change of mass must balance the flux of mass over all faces, so we have

$$\frac{\partial \rho}{\partial t} \Delta x \, \Delta y \, \Delta z = M_x^+ + M_x^+ + M_y^+ + M_y^+ + M_z^+ + M_z^+$$

Substitution of Taylor approximations and cleaning up gives

$$\frac{\partial \rho}{\partial t} = -\left(\rho \frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \rho}{\partial x}\mathbf{u}_x\right) - \left(\rho \frac{\partial \mathbf{u}_y}{\partial y} + \frac{\partial \rho}{\partial y}\mathbf{u}_y\right)$$
$$- \left(\rho \frac{\partial \mathbf{u}_z}{\partial z} + \frac{\partial \rho}{\partial z}\mathbf{u}_z\right)$$

Rewriting gives Continuity Equation

So using ∇ operator we can rewrite the equation,

$$\frac{\partial \rho}{\partial t} = -\left(\rho \frac{\partial \mathbf{u}_x}{\partial x} + \frac{\partial \rho}{\partial x}\mathbf{u}_x\right) - \left(\rho \frac{\partial \mathbf{u}_y}{\partial y} + \frac{\partial \rho}{\partial y}\mathbf{u}_y\right) - \left(\rho \frac{\partial \mathbf{u}_z}{\partial z} + \frac{\partial \rho}{\partial z}\mathbf{u}_z\right)$$

more compactly as follows,

$$\frac{\partial \rho}{\partial t} = -\nabla \rho \cdot \mathbf{u} - \rho \nabla \cdot \mathbf{u}$$

Now using the “product rule” backward gives the desired results,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Things to Remember

Special Cases got Special Names

Euler Equations or Inviscid Flow

- Zero viscosity
- Zero thermal conductivity (Not relevant for us as we only consider mass and momentum conservation in these slides)

For incompressible Newtonian fluids, The Euler equations take the form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = -\nabla p + \mathbf{F}$$
$$0 = \nabla\cdot\mathbf{u}$$

Stokes Flow or Creeping Flow or Creeping Motion

- For low Reynolds number $Re \ll 1$
- Inertia terms (both advection and unsteady term) are small compared to viscous forces

Hence, For incompressible Newtonian fluids, The Stokes equations take the form

$$\mathbf{0} = \mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{F}$$
$$0 = \nabla \cdot \mathbf{u}$$

Steady State Flow

- No time changes

For incompressible Newtonian fluids, the steady-state equations take the form

$$\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} = \mu\nabla^2\mathbf{u} - \nabla p + \mathbf{F}$$
$$0 = \nabla\cdot\mathbf{u}$$

Potential Flow

Assumes

- No rotation $\nabla \times \mathbf{u} = 0$ means that $\mathbf{u} = -\nabla\phi$, where ϕ is a scalar potential field.
- Inviscid, incompressible steady state flow (often used assumptions)

No compression $\nabla \cdot \mathbf{u} = 0$ results in the Laplace equation

$$\nabla^2\phi = 0$$

Assignment

What does $(\mathbf{u} \cdot \nabla)\mathbf{u}$ mean?

Assignment

Show in detail the mathematical derivation from the equation of conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

to the divergence-free condition given by the equation,

$$\nabla \cdot \mathbf{u} = 0\,.$$

Explain the arguments used for each step in the derivation.

Assignment

Deriving Navier Stokes equation from Taylor expansions

- Draw a small cube of fluid showing faces and coordinate axes
- For each of the six faces write up the corresponding Taylor terms. Pressure and velocity terms are on Page 43 and mass density terms on Page 44.
- For each face write up the outward unit normal vector.
- Explain the minus sign on M_x^+ from Page 46.
- Do the derivation on Page 47 in full detail on your own.

Assignment

Potential flow

- Show that for a potential flow we have $\nabla \times \nabla\phi = 0$. Potential flow is defined on Page 53.