Simplicial Complexes

A short Introduction to Algebraic Topology and Discrete Geometry

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The Definition of a Simplex

A simplex is defined as the point set consisting of the convex hull of a set of affinely independent points.

- Let $\{v_i\}^{n+1}$ denote a affinely independent point set containing $n+1$ points. Henceforth named the vertex set and its elements the vertices. The simplex, σ_n, is defined as the point set,

$$\sigma_n \equiv \left\{ v \mid v = \sum_{i=1}^{n+1} \lambda_i v_i, \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad 0 \leq \lambda_i \leq 1 \quad \forall i \right\}$$

- In three dimensional Euclidean space we can have up to four affinely independent vertices. This implies that $n = 0, 1, 2, 3$ are the only possible choices in a three dimensional space.

Affinely Independent Point Set

- Let the points $v_i \mathbb{R}^n$ for $i = 1$ to m be given
- Then the point set $\{v_i\}_1^m$ is affinely independent if the vector set $\{v_i - v_1\}_2^m$ is a linear independent set.
- A vector set $\{p_j\}_1^k$ is linear independent if and only if

$$0 = \alpha_1 p_1 + \cdots + \alpha_k p_k$$

when all scalar coefficients $\alpha_j = 0$ for all j.

The Vertex Set Operation

Let $\{v_i\}^{n+1}$ denote a affinely independent point set containing $n+1$ points and defining the point set of the simplex, σ_n.

$$\mathbf{vert}(\sigma_n) \equiv \{v_i\}^{n+1} = \{v_1, v_2, \ldots, v_{n+1}\}$$

As short-hand notation we use the labeling $\sigma_n \equiv \{v_1, v_2, \ldots, v_{n+1}\}$ as the notation that defines the simplex.

Simplex Dimension

The number of linear independent vertex basis vectors of the point-set of the simplex will be denoted as the dimension of the simplex. Thus we have

$$\mathbf{dim}(\sigma_n) \equiv n$$

for $n = 0, 1, 2, 3, 4$ and so on.

The Orientation of a Simplex

The orientation of a simplex is given by the ordering of the vertex set up to an even permutation (even number of two-element swaps)

- Thus, there exist only two classes of orientations

Example: given $\sigma_2 = \{v_1, v_2, v_3\}$ then

- $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_1\}$, and $\{v_3, v_1, v_2\}$ are of same orientation
- $\{v_2, v_1, v_3\}$, $\{v_1, v_3, v_2\}$, and $\{v_3, v_2, v_1\}$ are of same orientation

but $\{v_1, v_2, v_3\}$ and $\{v_2, v_1, v_3\}$ are of different orientations.

More on Orientations

- A zero-simplex (a single vertex) has no orientation
- The two different orientations are often designated by a sign, $+1$ or -1.
- One convention for picking an orientation is to use the determinant of the vertex basis,

$$\mathbf{sgn}(\sigma_n) \equiv \mathbf{sgn}(\mathbf{det}(\begin{bmatrix} (v_2 - v_1) & (v_3 - v_1) & \cdots & (v_{n+1} - v_1) \end{bmatrix}))$$

- If we are given the orientation $\mathbf{sgn}(\sigma_1) = \mathbf{sgn}(\{v_1, v_2\})$ then the opposite orientation is written as $\mathbf{sgn}(\{v_2, v_1\})$
- Or even more shorthand we use $\{v_1, v_2\} = -\{v_2, v_1\}$

Examples of Notation

So we have

- $$\mathbf{sgn}(\{v_1, v_2, v_3\}) = \mathbf{sgn}\left(\mathbf{det}\left(\begin{bmatrix} v_2 - v_1 & v_3 - v_1 \end{bmatrix}\right)\right)$$
- $$\mathbf{sgn}(\{v_1, v_2, v_3\}) = \mathbf{sgn}(\{v_2, v_3, v_1\} = \mathbf{sgn}(\{v_3, v_1, v_2\})$$
- $$\mathbf{sgn}(\{v_2, v_1, v_3\}) = \mathbf{sgn}(\{v_1, v_3, v_2\}) = \mathbf{sgn}(\{v_3, v_2, v_1\})$$
- $$\mathbf{sgn}(\{v_1, v_2, v_3\}) = -\mathbf{sgn}(\{v_2, v_1, v_3\})$$

A Simplex Face (Sub-simplex)

A face σ_m of a simplex σ_n is a simplex spanned by the subset of vertices of $\{v_i\}^{n+1}$

$$\mathbf{vert}(\sigma_m) \subseteq \mathbf{vert}(\sigma_n)$$

Observe

- Any face is itself a simplex
- By definition of a face any simplex is a face of itself.

If $\mathbf{dim}(\sigma_m) < \mathbf{dim}(\sigma_n)$ we call σ_m a proper face of σ_n.

The Boundary of a Simplex

We will define the boundary, Γ, of a simplex, σ to denote the set of faces having fewer elements in the vertex set than σ.

$$\Gamma(\sigma_n) \equiv \{\sigma_m \quad | \quad m < n \quad \wedge \quad \mathbf{vert}(\sigma_m) \subseteq \mathbf{vert}(\sigma_n)\}$$

Thus for $\sigma_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ (a triangle), the boundary is by our definition

$$\Gamma(\sigma_2) = \{\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}, \{\mathbf{v}_1\}, \{\mathbf{v}_2\}, \{\mathbf{v}_3\}\}$$

Thus, it is merely the edges and the vertices of the triangle.

The Boundary Operator

A slightly different definition defines the boundary operator of σ_n to be all faces having exactly $n-1$ elements in their vertex sets.

$$\partial(\sigma_n) \equiv \{\sigma_m \quad | \quad m = n-1 \quad \wedge \quad \mathbf{vert}(\sigma_m) \subseteq \mathbf{vert}(\sigma_n)\}$$

Usually, the orientations of the faces must be handled carefully.

- The boundary operator yields a set of $n+1$ simplexes $\partial(\sigma_n) = \{\sigma_m^j\}_{j=1}^{n+1}$ where

$$\sigma_m^j = (-1)^{j+1}\{v_1, \ldots, \hat{v}_j, \ldots, v_{n+1}\}$$

  and $\hat{v}_j$ means that v_j is dropped.

Observe that from a "geometric point set" viewpoint $\partial(\sigma_n) \equiv \Gamma(\sigma_n)$, only "topological set-wise" $\partial(\sigma_n) \neq \Gamma(\sigma_n)$.

The Closure of a Simplex

The closure operation is the union of the simplex and its boundary. Here is a the combinatorial notion of closure

$$\mathbf{cl}(\sigma_n) \equiv \sigma_n \cup \Gamma(\sigma_n)$$

Thus for $\sigma_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ we have

$$\mathbf{cl}(\sigma_n) = \{\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}, \{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}, \{\mathbf{v}_1\}, \{\mathbf{v}_2\}, \{\mathbf{v}_3\}\}$$

From a point-set viewpoint, one could just as easily have used the boundary operator $\partial(\sigma_n)$ in place of $\Gamma(\sigma_n)$ in the above definition.

The Closure of a Set of Simplexes

Given a set of simplexes $\mathcal{K} = \{\sigma^1, \ldots, \sigma^N\}$ the closure of $\mathcal{K}$ is defined as

$$\mathbf{cl}(\mathcal{K}) \equiv \bigcup_{\sigma^k \in \mathcal{K}} \mathbf{cl}(\sigma^k)$$

The Interior of a Simplex

We define the interior of a simplex as the point set of the simplex minus the points on the boundary.

$$\mathbf{int}(\sigma_n) \equiv \mathbf{cl}(\sigma_n) \setminus \Gamma(\sigma_n)$$

- Observe that all point sets are closed. Thus the vertices of a triangle are contained in the edges of the triangle and both the vertices and the edges of the triangle are contained in the triangle.

Observe from a point-set viewpoint we have $\mathbf{int}(\sigma_n) = \mathbf{cl}(\sigma_n) \setminus \partial(\sigma_n)$.

Adjacent Simplexes

Two simplexes σ^i and σ^j are said to be adjacent if and only if

- $\mathbf{dim}(\sigma^i) = \mathbf{dim}(\sigma^j)$
- and they share a common face

$$\sigma^k = \sigma^i \cap \sigma^j \neq \emptyset$$

- and the dimension of the common face is exactly one lower than the dimension of the simplexes

$$\mathbf{dim}(\sigma^k) = n - 1$$

  where $n = \mathbf{dim}(\sigma^i) = \mathbf{dim}(\sigma^j)$

We define the boolean binary relation $\mathbf{adj}(\sigma^i, \sigma^k)$ to be true if and only if σ^i and σ^j are adjacent simplexes and false otherwise.

The Simplicial Complex

A simplicial complex is a finite collection $\mathcal{K}$ of simplexes and the following two properties are always true

- Every face $\sigma^k \subset \sigma^j$ of each simplex $\sigma^j \in \mathcal{K}$ is also a simplex in $\mathcal{K}$
- Any intersection of two simplexes σ^i and σ^j from $\mathcal{K}$ is

$$\sigma^i \cap \sigma^j = \begin{cases} \emptyset \\ \sigma^k \in \mathcal{K} \end{cases}$$

The Star (one-ring) of a Simplex

Given $\sigma \in \mathcal{K}$ then the star operator is given by

$$\mathbf{star}(\sigma) \equiv \{\sigma_n | \sigma_n \in \mathcal{K} \wedge \mathbf{vert}(\sigma) \subseteq \mathbf{vert}(\sigma_n)\}$$

That is the set of all simplexes that σ is a face of.

- A top-simplex is defined as having $\mathbf{star}(\sigma) = \sigma$
- The dimension of a simplicial complex is equal to the highest dimension top simplex in the simplicial complex

The star operator is sometimes called the co-boundary operator.

The Discrete Manifold

An n-dimensional discrete manifold is an n-dimensional simplicial complex that satisfies

- For each simplex the union of all n-dimensional incident n-simplexes forms an n-dimensional ball
- or a half-ball if the simplex is on the boundary

Thus, each $n-1$-dimensional simplex has exactly two adjacent n-dimensional simplexes if not on the boundary and exactly one n-dimensional simplex otherwise.

That is It!

Questions?

Further Reading

- Siggraph Asia 2008 course notes: Discrete Differential Geometry: An applied Introduction. (Read Chapters 7 and 8)
- Marek Krzysztof Misztal, Deformable Simplicial Complexes, PhD Thesis, IMM, DTU, 2010

The Skeleton

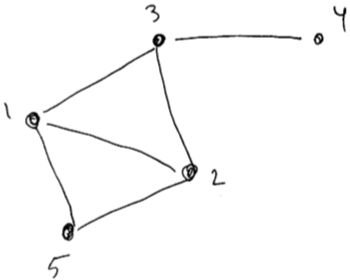
Given the simplicial complex $\mathcal{K}$ then the m-skeleton is given by

$$\mathcal{K}^{(m)} = \{\sigma_n | \sigma_n \in \mathcal{K} \land \mathbf{dim}(\sigma_n) = m\}$$

What Have We Learned?

- Geometry (=point-sets) and topology (= combinatorics) are two different things
- What we consider a nice mesh – the discrete manifold
- Star and link operators – are nice for making local changes
- Boundary and co-boundary operators are really useful for finite volume methods etc.

Assignment



$$\mathrm{sgn}(\{1,2,3\}) = ?$$

$$\mathrm{sgn}(\{1,2,5\}) = ?$$

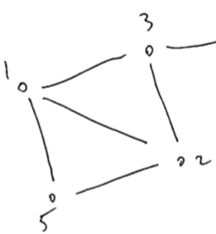
$$\partial(\{1,2,3\}) = ?$$

$$\partial(\{1,2,5\}) = ?$$

$$\partial(\{1,2\}) = ?$$

$$\partial(\{4\}) = ?$$

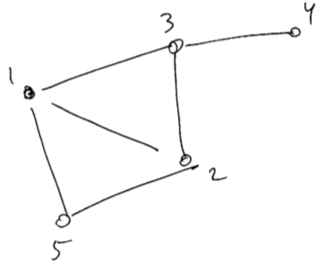
Assignment



$$K = \{ \{1,2,3\}, \{1,5,2\} \}$$

$$cl(K) = ?$$

Assignment



$$\mathrm{adj}(\{1,2,3\},\{1,5,2\}) = ?$$

$$\mathrm{adj}(\{1,2\},\{2,3\}) = ?$$

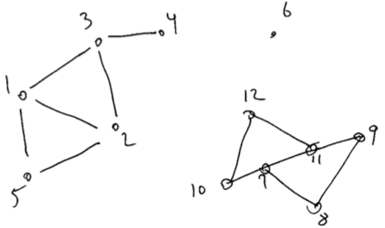
$$\mathrm{adj}(\{1,2,3\},\{3,4\}) = ?$$

$$\mathrm{adj}(\{3,4\},\{1,2\}) = ?$$

$$\mathrm{adj}(\{1\},\{2\}) = ?$$

Assignment

For each of the simplex collections below determine which are simplicial complexes



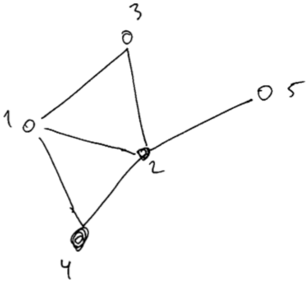
$$K_1 = \{ \{1,2,3\}, \{1,5,2\}, \{1,2\}, \{2,3\}, \{3,1\}, \{2,5\}, \{5,1\}, \{1\}, \{2\}, \{3\}, \{5\} \}$$

$$K_2 = K_1 \cup \{3,4\}$$

$$K_3 = K_1 \cup \{6\}$$

$$K_4 = \{\{7,8,9\}, \{10,11,12\}\} \cup \Gamma(\{7,8,9\}) \cup \Gamma(\{10,11,12\})$$

Assignment



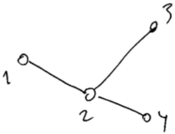
$K = \text{cl}(\{\{1,2,3\}, \{1,4,2\}, \{2,5\}\})$

$\text{star}(\{2\}) = ?$

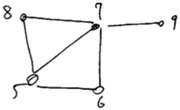
$\text{star}(\{1,2\}) = ?$

Assignment

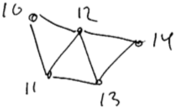
Determine which examples are discrete manifolds and which are not



$$K_1 = cl\left(\{\{1,2\},\{2,3\},\{3,4\}\}\right)$$



$$K_2 = cl\left(\{\{5,6,7\},\{5,7,8\},\{7,9\}\}\right)$$



$$K_3 = cl\left(\{\{10,11,12\},\{11,13,12\},\{12,13,14\}\}\right)$$

Assignment

- A Discrete Manifold is said to have consistent orientation if all top simplexes have the same orientation
- The link of a simplex $\sigma \in \mathcal{K}$ from a simplicial complex $\mathcal{K}$ is defined as

$$\textbf{link}(\sigma)) \equiv \textbf{cl}(\textbf{star}(\sigma)) \setminus \textbf{star}(\textbf{cl}(\sigma))$$

- Chains, Co-chains, and much more...